A Theoretical Model for the Penetration of a Shattered-Pellet Debris plume

Paul B. Parks

General Atomics

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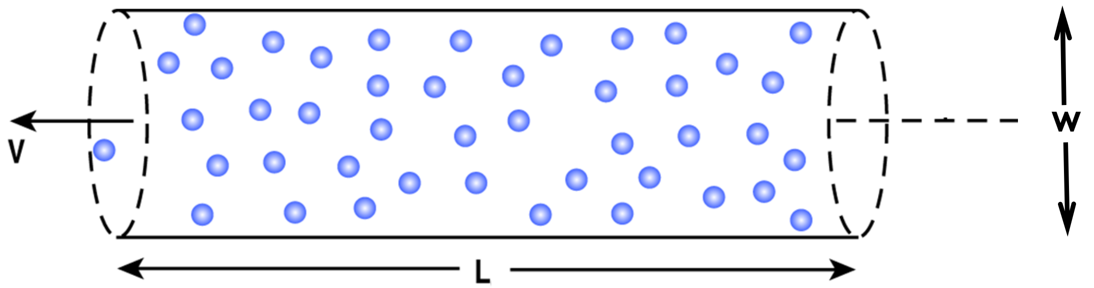
GENERAL ATOMICS

Shattered pellet fragments form a Debris Plume

- Simple rigid beam model: blunt cylindrical shape, uniformly distributed pellets all with same size and velocity V.



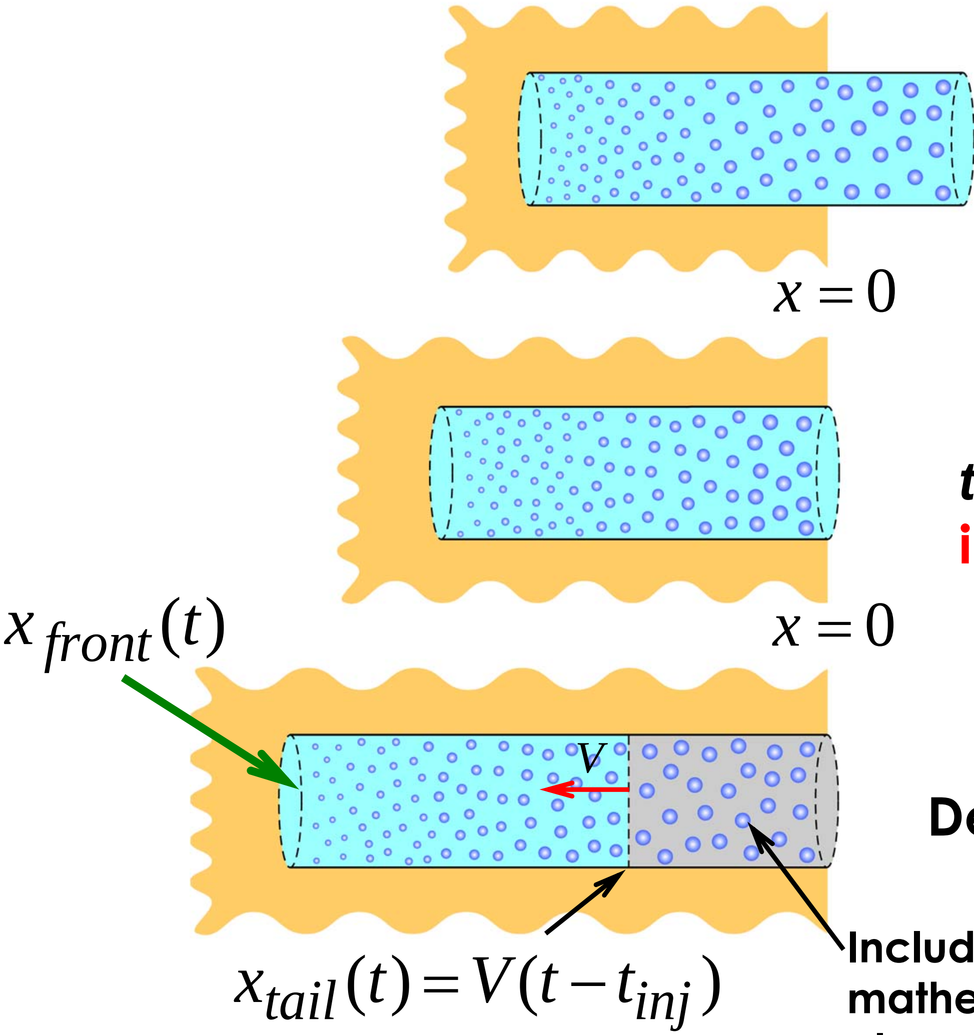
- SPI drift tube diameter in ITER is $D_{tube} = 4 \text{ cm}$

- Due to divergence, mean plume diameter downstream is larger, say $w = 30 \text{ cm}$

- Total Injection time from 2016 Debris Plume Theory $t_{inj} = 0.6 \text{ ms}$ for V =500 m/s

- Plume length $L = V \cdot t_{inj} = 30 \text{ cm}$ for for V =500 m/s

Stages of Propagation



Attached plume ($0 < t < t_{inj}$)

$t = t_{inj} = L/V$

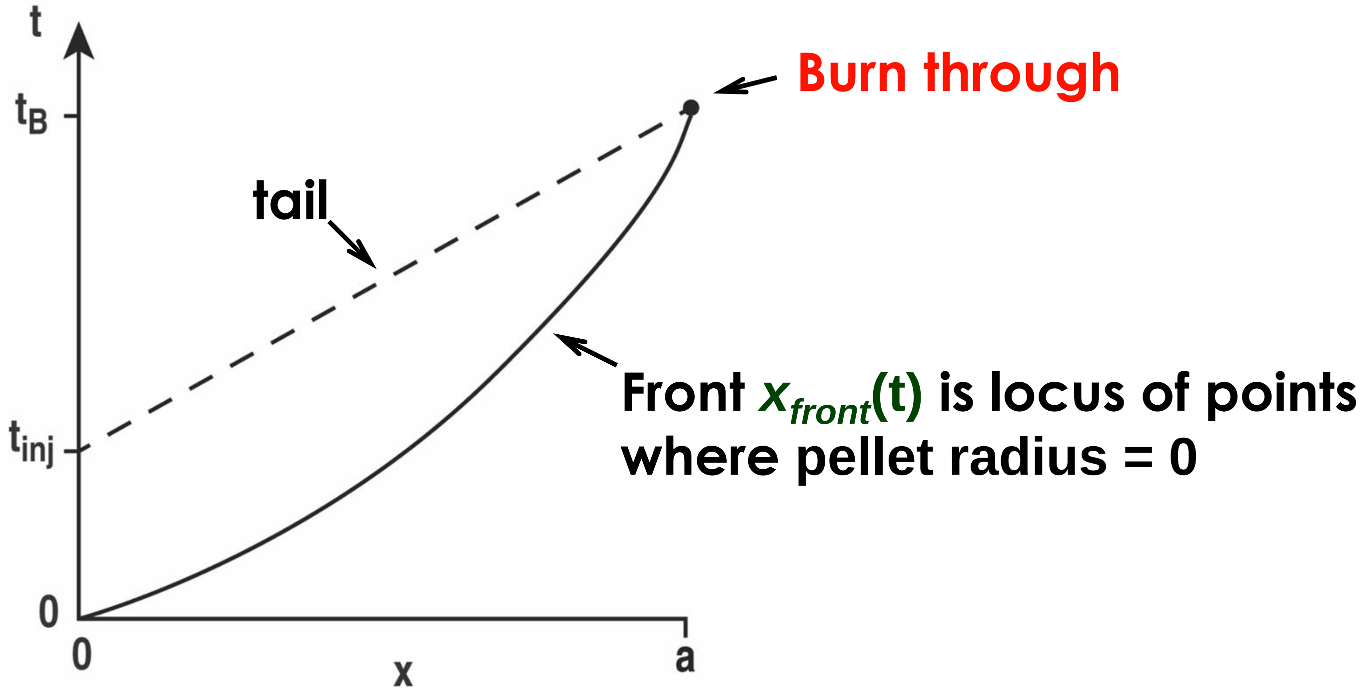
injection time

Detached stream ($t > t_{inj}$)

Include "virtual" section to ensure mathematically continuous BC at plasma edge

Find Trajectory of Moving Plume Front

When boring through plasma, the plume front moves slower than the original plume speed *V*, “pencil sharpening effect”.



- Ideal assimilation $x_{front} = a$, when the tail catches up with the front at the magnetic axis giving burnthrough time

$$t_B = t_{inj} + a/V$$

Kinetic Model of Ambient Plasma Cooling

- **Pellets ablate and deposit cold ionized ablation trail which expands along magnetic field and radiates.**
- **The ionized ablation material is tenuous enough to allow inter penetration of hot ambient plasma electrons Proof!**
  - **Columnar density of the ionized impurities remains constant while expanding along the magnetic field**

$$\Sigma_{\|} = \int_{-\infty}^{\infty} n_I \, ds = \text{constant}$$

  - **If all pellet fragments ablate fully such that impurities are distributed evenly across minor radius then from mass conservation** $\Sigma_{\|} = N_I / wak$

$$N_I = \text{number of neon atoms deposited, } k = \text{ number of injectors}$$

  - **Electrons streaming through plume suffer only a small collisional energy loss**

$$\frac{\Delta E}{E} = \frac{\Sigma_{\|}}{E} L(E) << 1 \quad L(E) = \frac{2\pi e^2 Z_a}{E} \ln\left[\frac{E}{I_*}\left(\frac{e}{2}\right)^{1/2}\right] \quad \text{(Bethe stopping power, } I_* = 135.5 \text{ eV for Ne)}$$

  - **This means pellet fragments are bathed in a two-temperature plasma: Hot ambient electrons and freshly ionized cold electrons. Only hot electrons do the ablating. How fast do they cool?**

Ambient Plasma Cooling Contd

- **Kinetic equation describes evolution of plasma electron distribution function due to inelastic collisions with impurity atoms/ions)**

$$\frac{\partial f}{\partial t} = \frac{\partial}{\upsilon^2 \partial \upsilon}\left(\upsilon^2 a_{drag} f\right) \qquad a_{drag} = \frac{\langle n_I \rangle}{m_e} L(E) \qquad E = m_e \upsilon^2 / 2$$

$\langle n_I \rangle$ = flux - surface - averaged impurity nuclei density

- **Pellet ablation rate depends on electron temperature of a Maxwellian plasma. Assume bulk electrons remain roughly Maxwellian:**

$$f(\upsilon) \approx f_{\max} = n_e \left(\frac{m_e}{2\pi T_e(t)}\right)^{3/2} \exp\left(-\frac{m\upsilon^2}{2T_e(t)}\right)$$

- **Take energy moment of kinetic equation to get**

$$\left.\frac{\partial T_e}{\partial t}\right)_x = -5.812 \times 10^{-6} \frac{\langle n_I \rangle}{T_e^{1/2}} Z_a \ln\left(\frac{T_e}{1.528 I_*}\right)$$

- **Use simpler approximation and generalize to neon-deuterium mixtures**

$$\left.\frac{\partial T_e}{\partial t}\right)_x = -2.2 \times 10^{-6} \frac{\langle n_{Ne} \rangle}{T_e^{1/4}} \left(Z_{Ne} + \frac{2X}{1-X}\right) \qquad X = \frac{\text{mol } D_2}{\text{mol } N_e + \text{mol } D_2}$$

Independent Pellet Ablation Model

- **Each pellet ablates as though it were isolated from the rest**
  - **Obscuration of || -electron flux by its fellow fragments is typically small**

$$\Delta q_{||} / q_{||} = \tau_{\text{pell}} << 1$$

number concentration of pellets

- **Optical depth** $\tau_{\text{pell}} = n_{\text{pell}} \pi r_p^2 w$

$$n_{\text{pell}} = \frac{\text{Total Mass added}}{\text{Mass per pellet}}$$

$$\tau_{\text{pell}} = \frac{3 \cdot \text{SDP}}{4 \rho_0(X) r_p}$$

Analogous to τ_{cloud} for scattering of sunlight by cloud water droplets, replacing LWP → SDP

Solid Debris Path (g/cm^2)

$\rho_0(X)$: **pellet density (g/cm^3)**

- **High level of solid pellet transparency (even more transparent than gas)**

$$N_{Ne} = 0.041 \text{ moles}, \; k = 2, \; L = 30 \text{ cm}, \; w = 30 \text{ cm}, \; r_p = 0.1 \text{ cm}, \; X = 0, \; \rho_0 = 1.444$$

$$\text{SDP} = 0.0004 \text{ g/cm}^2 \longrightarrow \tau_{\text{pell}} = 0.0024$$

A Practical Expression for the Ablation Rate of Composite Neon-Deuterium Pellets [§]

$$G = \lambda(X)\left(\frac{T_e}{2000}\right)^{5/3}\left(\frac{r_p}{0.2}\right)^{4/3} n_{e14}^{1/3}$$

G(g/s) T_e(eV)

r_p(cm) $n_e(10^{14}\text{cm}^{-3})$

$$\lambda(X) = 27.0837 + \text{Tan}[1.48709X]$$

- **Molar deposition rates per pellet**

$$\frac{dN_{Ne}}{dt} = \frac{(1-X)G}{W_{Ne}(1-X)+XW_{D_2}} \qquad \frac{dN_{D_2}}{dt} = \frac{XG}{W_{Ne}(1-X)+XW_{D_2}}$$

$$W_{Ne} = 20.183 \quad W_{D_2} = 4.0282 \quad \text{(g/mol)}$$

- **Pellet surface recession speed** $\dot{r}_p = -G/(4\pi r_p^2 \rho_0)$

$$\frac{Dy}{Dt} = -3.572\times10^{-6}\frac{\lambda(X)}{r_0^{5/3}\rho_0(X)} T_e^{5/3} n_{e14}^{1/3} \qquad y = (r_p/r_0)^{5/3}$$

§ Parks, to be published

Flux-Surface-averaged gas density build up

- **Build up rate of neon atoms on a magnetic flux shell of differential thickness** δx_ψ

$$\delta \dot{N}_{Ne} = n_{pell} \cdot \frac{dN_{Ne}}{dt} \cdot w^2 \delta x_\psi$$

  - **Flux shell volume** $\delta V_\psi = 2\pi R \cdot 2\pi r \delta x_\psi$

- **Flux-averaged neon density increase** $\frac{\partial \langle n_{Ne} \rangle}{\partial t} = \frac{\delta \dot{N}_{Ne}}{\delta V_\psi}$

→ $$\frac{\partial \langle n_{Ne} \rangle}{\partial t} = \frac{N_{Ne} A}{4\pi^2 L R r} \left(\frac{3G}{4\pi r_0^3 \rho(X)} \right) \qquad \frac{\partial \langle n_D \rangle}{\partial t} = \frac{2X}{1-X} \frac{\partial \langle n_{Ne} \rangle}{\partial t}$$

A: Avogadro's number

Coupled System of PDEs Describes 1-D SPI Dynamics

- **Independent variables** (ξ, t) $\quad \xi = x/a = \text{streamwise distance}$

- **Pellet radii change**

$$\frac{\partial y}{\partial t} + \frac{V}{a}\frac{\partial y}{\partial \xi} = -\frac{\Theta^{5/3} n_{e14}^{1/3}(\xi)}{t_{abl}}$$

- **Plasma Cooling**

$$\frac{\partial \Theta^{5/4}}{\partial t} = -\frac{\tilde{n}}{t_{cool}}$$

- **Flux-averaged neon density rise**

$$\frac{\partial \tilde{n}}{\partial t} = \frac{9}{5}\frac{y^{4/5}\Theta^{5/3} n_{e14}^{1/3}(\xi)}{(1-\xi)t_{abl}}$$

$$y(\xi,t) = \left(r_p(\xi,t)/r_0\right)^{5/3}$$

$$\Theta(\xi,t) = T(\xi,t)/T_0$$

$$\tilde{n}(\xi,t) = \langle n_{Ne} \rangle / n_{\max}$$

$$n_{\max} = \frac{N_{Ne}A}{4\pi^2 LRa}, \quad T_0 = T(a,0)$$

- **Characteristic time constants:**

$$t_{abl} = 2.8\times10^5 \left(\frac{r_0}{T_0}\right)^{5/3} \frac{\rho_0(X)}{\lambda(X)}$$

(Ablation time)

$$t_{cool} = 3.63\times10^5 \frac{T_0^{5/4}}{n_{\max}} \left(Z_{Ne} + \frac{2X}{1-X}\right)^{-1}$$

(Cooling time)

Insight From a 0-D Semi-Analytical Solution

- **Assume Plume L = a is deposited in plasma instantly at t = 0** $\partial/\partial\xi = 0$

$$\tilde{n} = 1 - y^{9/5} \qquad \Theta = \left[1 - d\left(1 - \frac{14}{9}y + \frac{5}{9}y^{14/5}\right)\right]^{12/35}$$

$$d = \frac{3t_{abl}}{2t_{cool}} \propto \frac{N_{Ne} r_0^{5/3}}{T_0^{35/12}}$$

$$\frac{dy}{dt} = -\frac{\omega(y)}{t_{abl}}, \qquad \omega(y) = \left[1 - d\left(1 - \frac{14}{9}y + \frac{5}{9}y^{14/5}\right)\right]^{4/7}$$

- **"Super-critical" injection d > 1: Plasma Cooling is so fast that temperature quench happens before pellets fully ablate**

$$\Theta \to 0\ , \quad y \to y_{crit}, \quad \tilde{n} \to 1 - y_{crit}^{9/5}, \quad t_{quench} \to t_{abl} \int_{y_{crit}}^{1} \omega(y)^{-1}\, dy$$

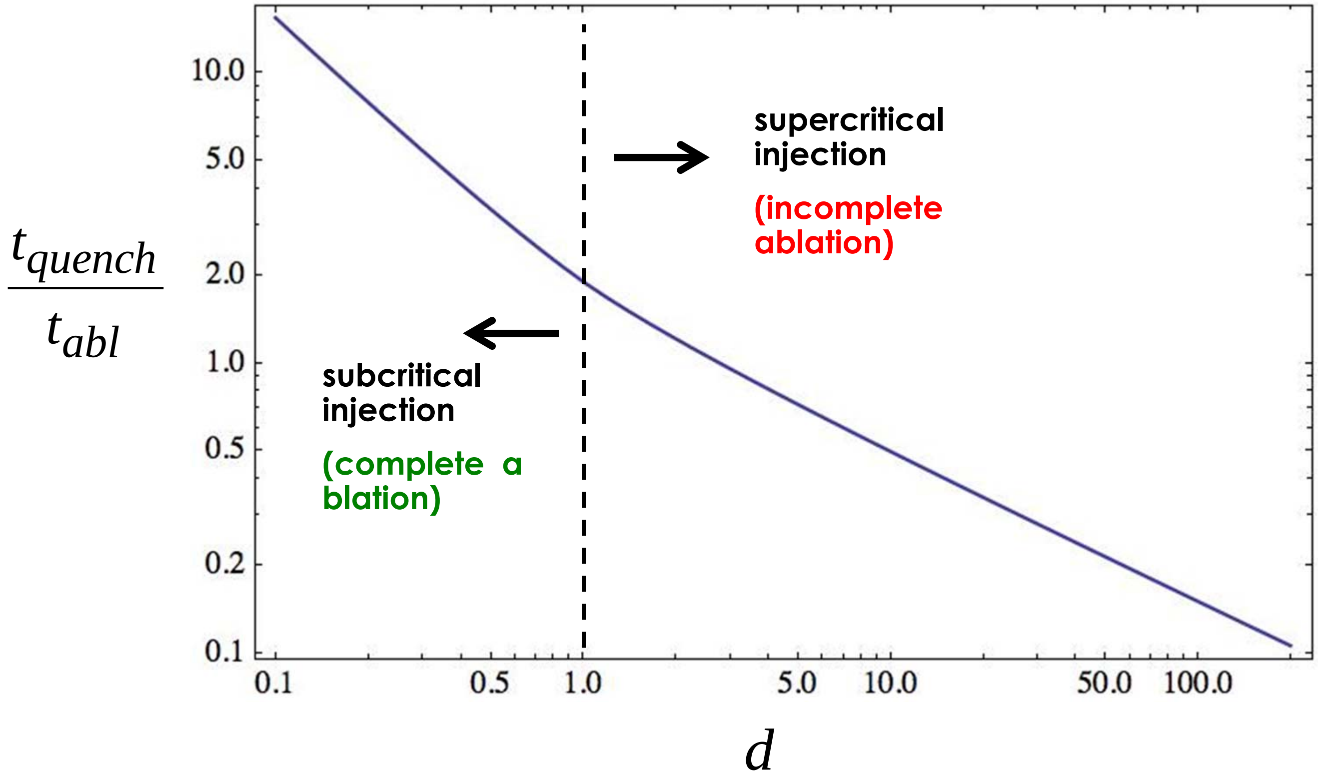
$$1 - (14/9)y_{crit} + (5/9)y_{crit}^{14/5} = d^{-1}$$

- **"Sub-critical" injection d < 1: Pellets "burn out" before temperature quench**

$$\text{STAGE 1:} \quad \Theta \to (1-d)^{12/35}\ , \quad y \to 0, \quad \tilde{n} \to 1 \qquad t \to t_* = t_{abl} \int_0^1 \omega(y)^{-1}\, dy$$

$$\text{STAGE 2:} \quad \Theta \to 0\ , \qquad t_{quench} \to t_* + t_{cool}(1-d)^{3/7}$$

Quench Time Versus d



$$d = \frac{3t_{abl}}{2t_{cool}} \propto \frac{N_{Ne} r_0^{5/3}}{T_0^{35/12}}$$

- **Examples:** $X = 0$ (pure neon), $r_{p0} = 0.15$ cm, $N_{Ne} = 0.041$ moles

$t_{abl} = 0.136$ ms and $d = 1.676$ and $t_{quench} = 0.183$ ms for $T_0 = 10$ keV

$t_{abl} = 0.0693$ ms and $d = 0.514$ and $t_{quench} = 0.228$ ms for $T_0 = 15$ keV

Transformation to 1-D Lagrangian Variables

$(x,t) \rightarrow (x,q) \quad q = t - \frac{x}{V}$

Lagrange coordinate q labels elemental slice of debris plume in motion

- **The "first arrivals" enter the plasma at t = 0 and x = 0 with Lagrangian label** $q = 0$

- **The tail pellets enter last with Lagrangian label** $q = t_{inj}$

- **Normalized coordinates**

$$(x,q) \rightarrow (\xi,\zeta) \qquad \xi = x/a \quad (0<\xi<1)$$
$$\zeta = q/t_{inj} \quad (0<\zeta<1)$$

- **Additional time scales**

$$t_{inj} = \frac{L}{V}, \quad t_{transit} = \frac{a}{V}$$

Transformed Equations Leads to a Cauchy-like Problem

- **Pellet radii change** $\left.\frac{\partial y}{\partial \xi}\right)_{\zeta} = -\sigma_1 \tilde{\Theta}^{4/3}$ $\sigma_1 = \frac{t_{transit}}{t_{abl}}$

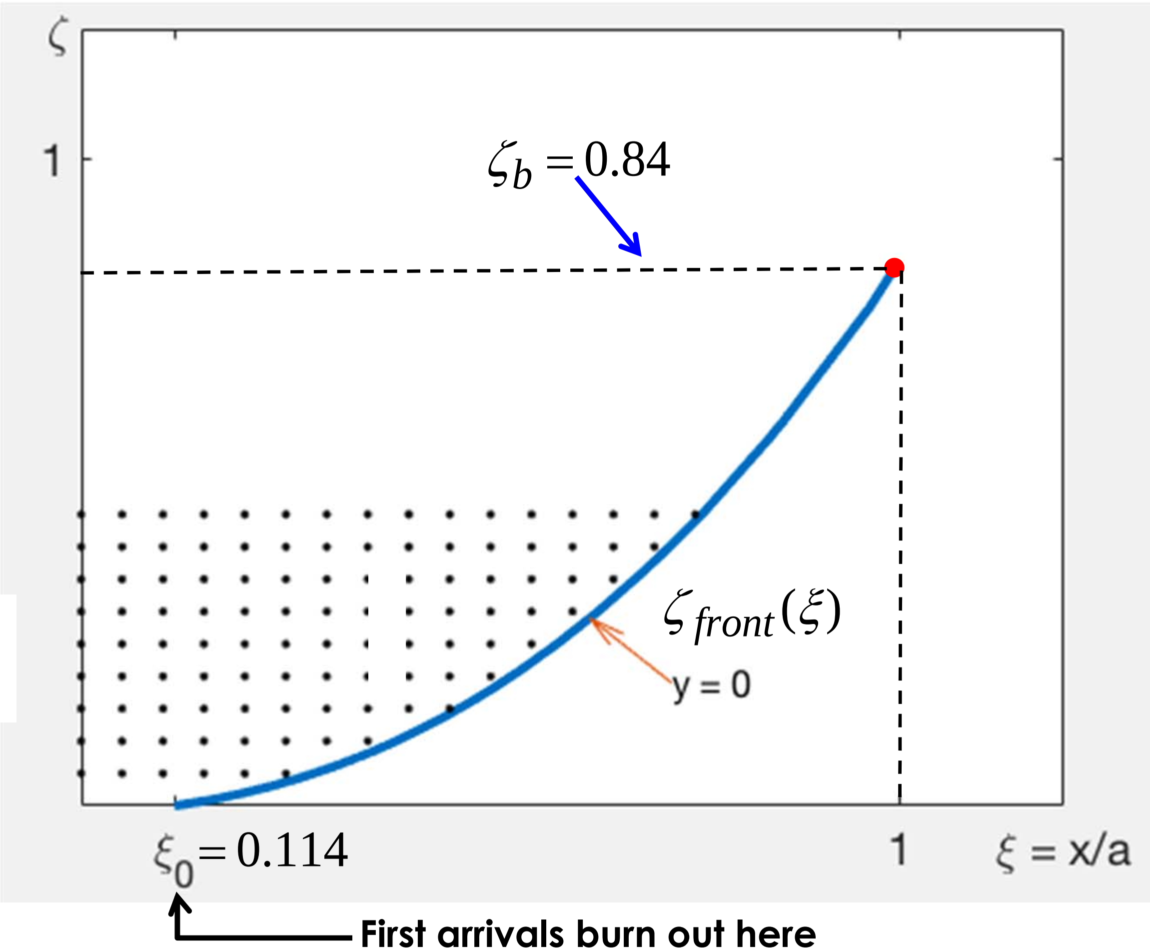
- **Plasma Cooling** $\left.\frac{\partial \tilde{\Theta}}{\partial \zeta}\right)_{\xi} = -\sigma_2 \tilde{n}$ $\sigma_2 = \frac{t_{inj}}{t_{cool}}, \quad \tilde{\Theta} = \Theta^{5/4}$

- **Flux-averaged neon density rise** $\left.\frac{\partial \tilde{n}}{\partial \zeta}\right)_{\xi} = \sigma_3 \frac{y^{4/5} \tilde{\Theta}^{4/3}}{(1-\xi)}$ $\sigma_3 = \frac{9 t_{inj}}{5 t_{abl}}$

- **In these equations we assumed a flat density profile with** $n_{e14}(\xi) = 1$

- **Cauchy data:** **along the** ζ**–axis** $y = 1$
  along the ξ**–axis** $\tilde{n} = 0$ and $\tilde{\Theta} = \left(T_e(\xi)/T_0\right)^{5/4}$ (↑ **initial T_e profile**)

Numerical Solution of Front Trajectory in Hodograph Plane



The front trajectory intercepts the magnetic axis with

$$\zeta = \zeta_b = 0.84$$

This means 84% of solid debris plume was annihilated, with 16% left only partially ablated

The time for ζ_b element to reach the magnetic axis is

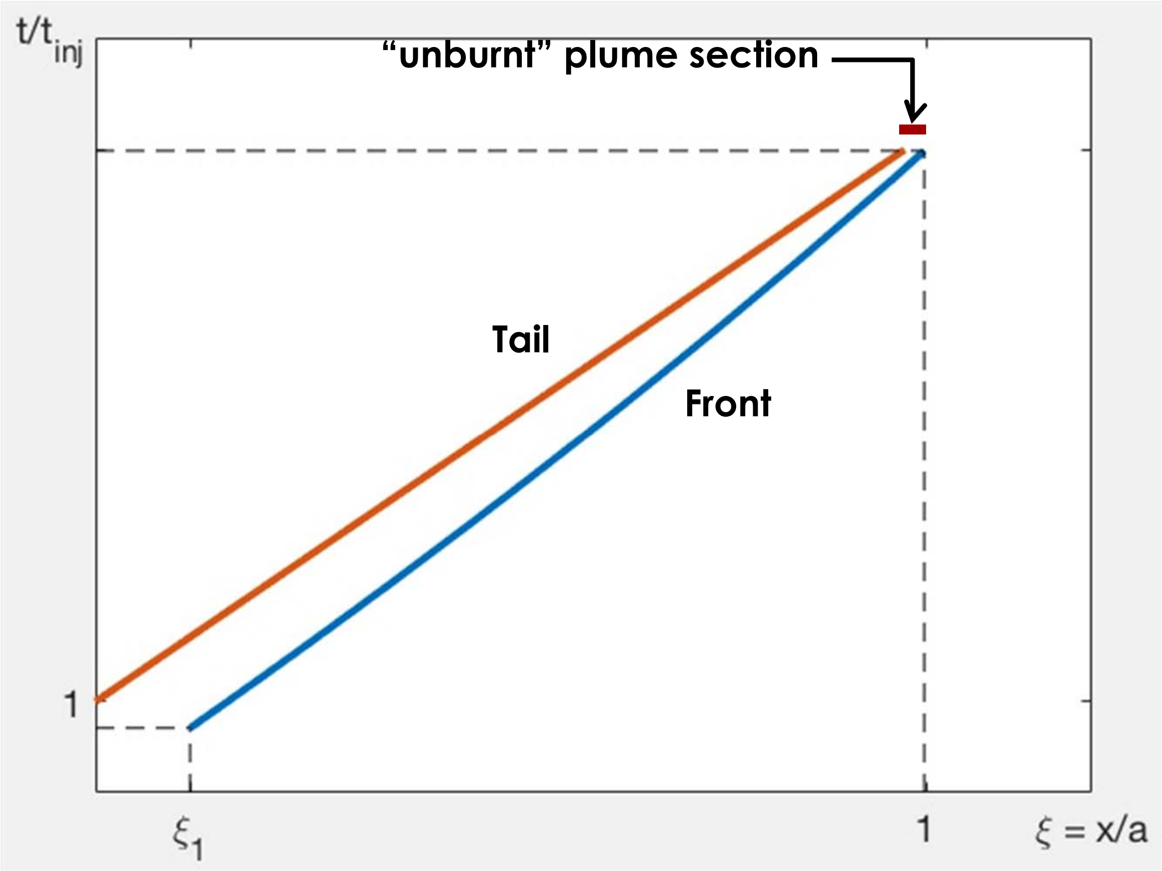
$$t_b = \frac{a}{V} + \zeta_b t_{inj} = 4.24 \text{ ms}$$

For $t > t_b$ the surviving plume elements cross magnetic axis with no further ablation

- **Parameters** $N_{Ne} = 0.041$ moles, $X = 0.5$, $r_{p0} = 0.15$ cm, $T_0 = 19.5$ keV, $L = 30$ cm, $a = 187$ cm, $V = 500$ m/s, $t_{inj} = 0.6$ ms.

Numerical Solution of Front Trajectory in (x,t) Plane

- Front has smaller velocity than tail velocity V due to erosion (pencil sharpening effect)



- To assimilate entire plume when front hits magnetic axis we could reduce d

$$d \propto \frac{N_{Ne} r_0^{5/3}}{T_0^{35/12}}$$

- e.g. make smaller pellets, or
- Reduce V or L
- Numerical iteration

Dilution Cooling Model

- **Assume most electrons added are free (valid for lots of deuterium X~1**
  - **Pellet radii change** $$\frac{\partial y}{\partial t}+V\frac{\partial y}{\partial x}=-\frac{\Theta^{5/3}\tilde{n}^{1/3}}{t_{abl}}$$
  - **Flux-averaged electron density rise** $$\frac{\partial \tilde{n}}{\partial t}=\frac{9}{10}\frac{a}{L}\frac{\Delta n_e}{n_{e0}}\frac{y^{4/5}\Theta^{5/3}\tilde{n}^{1/3}}{\rho t_{abl}}$$

$$\tilde{n}=\frac{n_e(x,t)}{n_{e0}},\quad \rho=1-x/a \qquad t_{abl}=\frac{2.8\times10^5}{n_{e014}^{1/3}}\left(\frac{r_0}{T_0}\right)^{5/3}\frac{\rho_0(X\approx1)}{\lambda(X\approx1)}$$

$$\Delta n_e=\frac{\text{added free electrons}}{\text{plasma volume}}$$

- **Dilution Cooling** $\Theta(x,t)\tilde{n}(x,t)=P(x)$ $\qquad P(x)=\frac{p_e(x)}{p_{e0}}$ normalized pressure
- **Eliminate** $\Theta(x,t)$ **from equations**

Dilution Cooling Model cont'd

$$\tau = t/t_*, \quad s = x/Vt_*, \quad Z = \left(\frac{\tilde{n}}{\tilde{n}_*}\right)^{7/3} \qquad t_* = t_{abl}\tilde{n}_*^{4/3} \qquad \tilde{n}_* = \frac{21}{10}\frac{a}{L}\frac{\Delta n_e}{n_{e0}}$$

- **Reduced Equations**

$$\frac{\partial y}{\partial \tau} + \frac{\partial y}{\partial s} = -Z^{-4/7}P(s)^{5/3}$$

$$\frac{\partial Z}{\partial \tau} = y^{4/5}P(s)^{5/3}/\rho(s)$$

- **Convert to Lagrangian variables** $(s,\tau) \rightarrow (s,\varsigma) \quad \varsigma = \tau - s$

$$\left.\frac{\partial y}{\partial s}\right)_{\varsigma} = -Z^{-4/7}P(s)^{5/3}$$

$$\left.\frac{\partial Z}{\partial \varsigma}\right)_{s} = y^{4/5}P(s)^{5/3}/\rho(s)$$

Further Simplifying Transformations

$$\mathrm{H} = \frac{Z\rho}{P^{5/3}} \qquad u(s) = \int_0^s P(s')^{5/7} \rho(s')^{4/7} \, ds' \quad (= 0 \text{ at plasma boundary } s = 0)$$

- **With above definitions we get**

$$\left.\frac{\partial y}{\partial u}\right)_{\varsigma} = -\mathrm{H}^{-4/7} \qquad \left.\frac{\partial \mathrm{H}}{\partial \varsigma}\right)_{u} = y^{4/5}$$

- **Similarity variable** $\eta = \dfrac{\varsigma}{u^{7/4}}$ **converts PDEs to ODEs**

with $\Phi = \dfrac{\mathrm{H}}{u^{7/4}}$

$$\frac{dy}{d\eta} = \frac{4}{7\eta}\Phi^{-4/7} \quad \text{pellet radii}$$

$$\frac{d\Phi}{d\eta} = y^{4/5} \quad \text{density rise}$$

Universal Solution for y and Φ

- **Dependent variables are reduced pellet radii and electron density**

$$y = \left(\frac{r_p}{r_0}\right)^{5/3} \qquad \Phi = \frac{\mathrm{H}}{u(s)^{7/4}} = \frac{\rho}{P(\rho)^{5/3} F(\rho)^{7/4}} \left(\frac{Vt_*}{a}\right)^{7/4} \left(\frac{\tilde{n}}{\tilde{n}_*}\right)^{7/3}$$

Where $u(s)\left(\frac{Vt_*}{a}\right) = F(\rho) = \int_\rho^1 P(\rho')^{5/7} \rho'^{4/7} \, d\rho' \quad (= 0 \text{ at plasma boundary } \rho = 1)$

- **Boundary Conditions:**

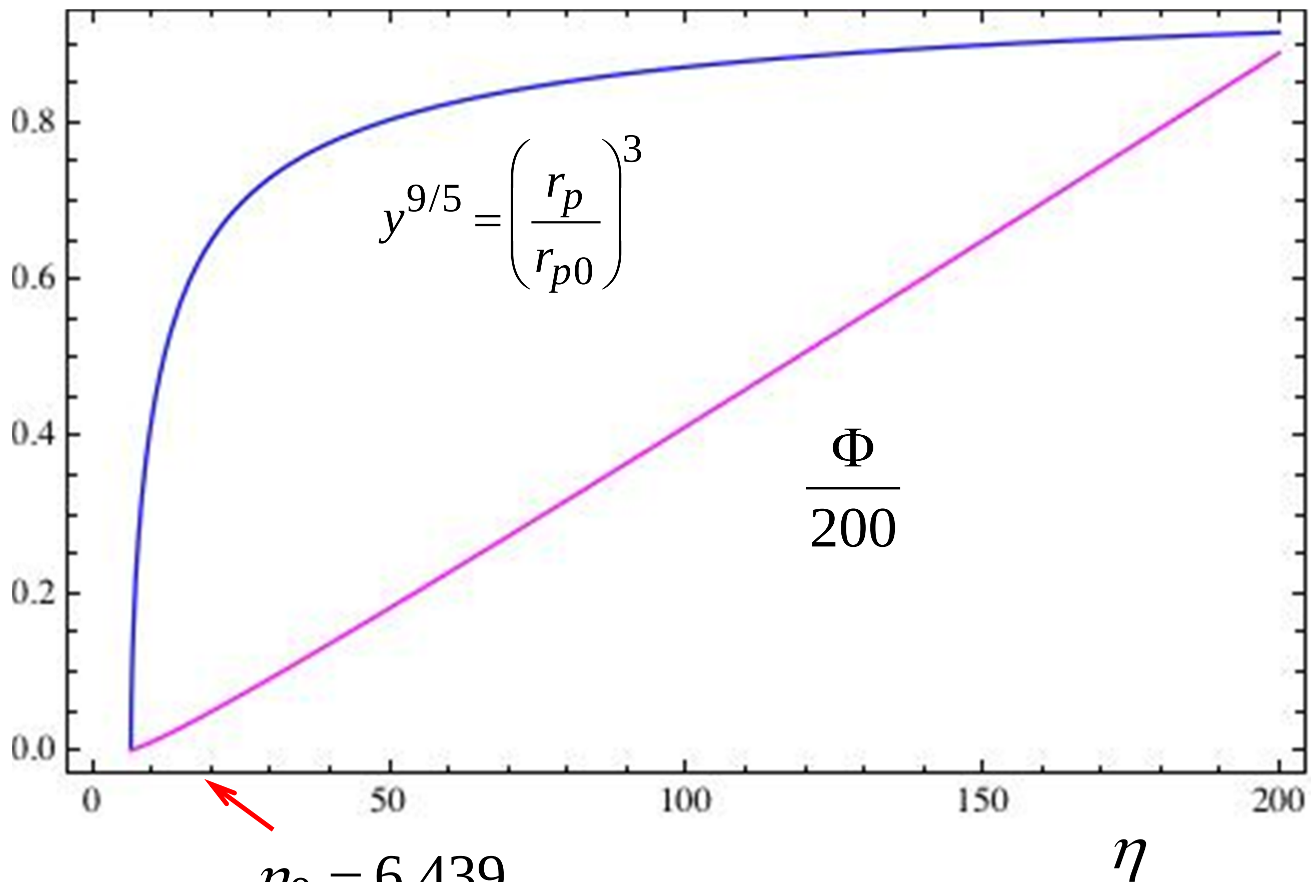
✓ $y = 1$ at plasma boundary $\eta = \varsigma / u^{7/4} \to \infty$

✓ $y = 0$ pellet radii = 0 at moving front $\eta = \eta_0$

✓ $\Phi = 0$ added density = 0 at moving front $\eta = \eta_0$

- **Isn't that 3 boundary conditions? No. Only 2 because the front position η_0 is not known *a priori*. We can only know η_0 by using a shooting method**

Plot of Universal Solutions



$\eta_0 = 6.439$

(numerical value of plume front position)

Solution for Trajectory of Moving Debris Front

- **Front trajectory**

$$x_{front} = Vt - \eta_0 \frac{a^{7/4}}{(Vt_*)^{3/4}} F(1 - x_{front} / a)^{7/4}$$

- **Tail trajectory**

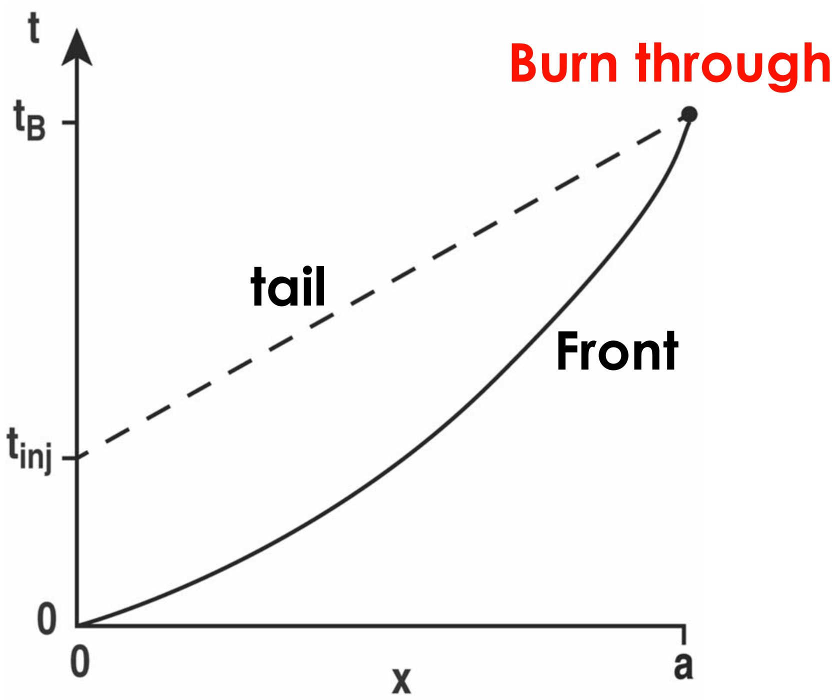
$$x_{tail} = V(t - t_{inj})$$

- **Optimized injection: Burn through when front and tail meet at the magnetic axis**

$$x_{tail} = x_{front} = a$$

$$t_B = t_{inj} + a / V$$

- **This leads to the optimum velocity…**



Optimum Velocity

- **More added mass → more self-cooling → lower velocity**

$$V = \left(\frac{10\eta_0}{21}\right)^{4/3} F(0)^{7/3} \frac{a}{t_{abl}} \left(\frac{n_{e0}}{\Delta n_e}\right)^{4/3}$$

$$t_{abl} = \frac{2.8\times 10^5}{n_{e014}^{1/3}} \left(\frac{r_0}{T_0}\right)^{5/3} \frac{\rho_0(X)}{\lambda(X)}$$

- **In ITER with** $\Delta n_e / n_{e0} = 30$ $\quad T_{e0} = 19.5\,\text{keV},\ a = 1.87\,\text{m},\ r_0 = 2\,\text{mm},\ F(0) = 0.34$

→

$$V = 1037\ \text{m/s}\ \ \text{for}\ X = 1\ \ (\text{pure}\ D_2)$$
$$V = 576\ \text{m/s}\ \ \text{for}\ X = 0.9\ \ (\text{mostly}\ D_2)$$
$$V = 210\ \text{m/s}\ \ \text{for}\ X = 0.5$$

- **$E_{critical}$ ~ 5V/m , E_{eff} ~ 10V/m , runaway beam decay time ~ 200 ms**

Final Density Profile for Optimized Injection

- **Space-time electron density profile evolution**

$$n_e(x,t) = \frac{21\Delta n}{10\eta_0 F(0)} \frac{P^{5/7}(\rho)}{\rho^{3/7}} \left(\frac{F(\rho)}{F(0)}\right)^{3/4} \Phi^{3/7}\left[\eta_0\left(\frac{t - x/V}{t_{inj}}\right)\left(\frac{F(\rho)}{F(0)}\right)^{-7/4}\right]$$

For $t < t_{inj}$ $\quad 0 < x < x_{front}(t)$ $\quad$ and for $\quad t > t_{inj}$ $\quad x_{tail}(t) < x < x_{front}(t)$

- **Density profile is "frozen in time" for $0 < x \le x_{tail}(t)$. So final density profile obtained after burnout is found by setting $x = x_{tail}(t)$ in above expression:**

$$n_{efinal}(\rho) = \frac{21\Delta n_e}{10\eta_0 F(0)} \frac{P^{5/7}(\rho)}{\rho^{3/7}} \left(\frac{F(\rho)}{F(0)}\right)^{3/4} \Phi^{3/7}\left[\eta_0\left(\frac{F(\rho)}{F(0)}\right)^{-7/4}\right] \qquad \Delta n_e = \frac{N_e}{2\pi R\pi a^2\kappa}$$

- **Check for consistency: Does** $N_e = 4\pi R\kappa a^2 \int_0^1 n_{efinal}\,\rho\, d\rho$ **?**

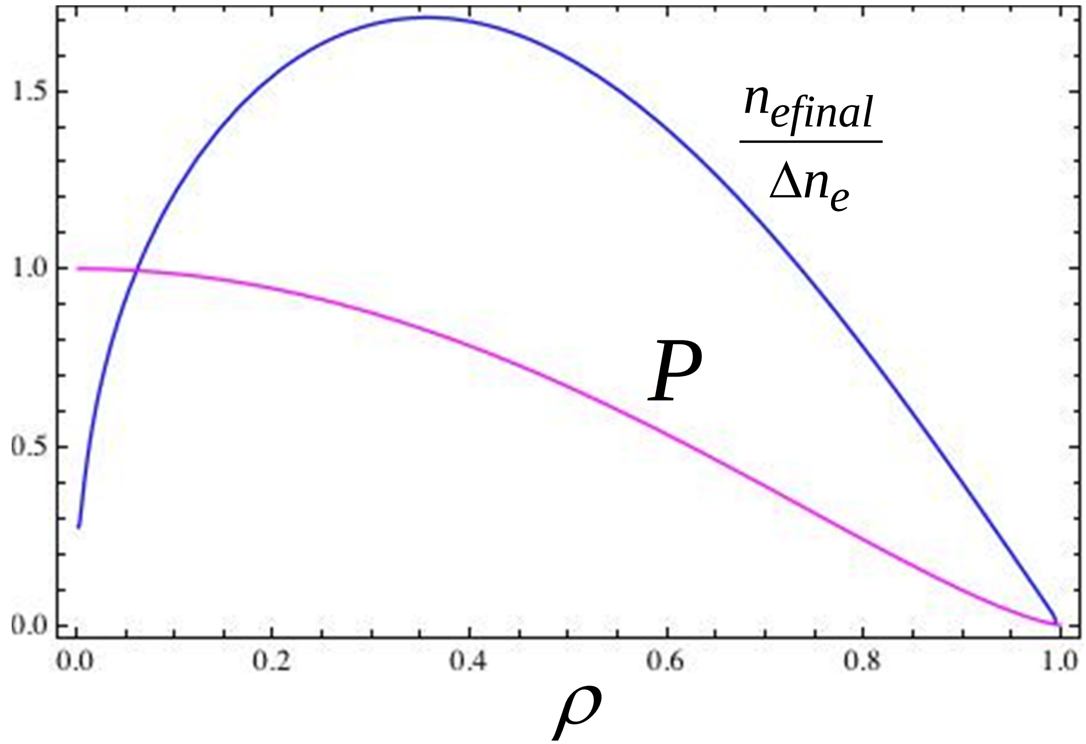
$$1 = \frac{12}{5}\int_{\xi_0}^{\infty} \frac{Z^{3/7}(\phi)}{\phi^2} d\phi$$

YES, equality holds true when $\xi_0 = 6.439$ ✓

Plot of Final Density Profile

- **Using a special normalized pressure profile** $P(\rho) = (1-\rho^2)^{7/5}$

$$F(\rho)/F(0) = 1 - \left(\frac{25}{14}\rho^{11/7} - \frac{11}{14}\rho^{25/7}\right) \quad \text{with} \quad F(0) = \frac{98}{275} = 0.356$$



- **For optimum injection, the added density profile is skewed towards the magnetic axis** $\rho = 0$

Summary and Future Directions

- Developed a 1-D analytic model for the penetration of SPI plume in a plasma which includes plasma cooling by the ablated gas trail
    - two cooling models: kinetic based for neon-D2 and dilution cooling for mainly D2.
    - will compared results with NIMROD that assume dilution cooling with ion Te = Ti

- Publish Z > 1 pellet ablation models and SPI theory

- Improve hot tail RE burst physics model for realistic SPI and pellet injection situations

- Extend SPI model to 2-D geometry important for optically thick gas

- Explore alternative particle injection approaches such as Be shell pellets