Linear Stability of a Fluid Runaway Electron Beam

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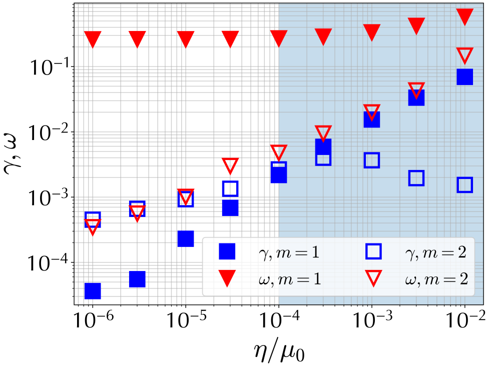
Preview

- Numerical analysis of the linearzied MHD+fluid runaway electron(RE) equations in cylindrical geometry was performed.
- Resistivity scaling of linear growth rates for a tearing unstable equilibrium shows the fastest growing mode transitions from $m = 2$ to $m = 1$ at large resistivity.
- The mode is overstable, with $|\omega| \gg \gamma$.
- Preliminary analysis suggests a resistive hose-like instability[a][b]
- Dominance at high resistivity suggests the beam mode may be important in post-thermal quench tokamak scenarios.



[a] M. N. Rosenbluth, Physics of Fluids **3**, 932–936 (1960).

[b] S. Weinberg, Journal of Mathematical Physics **8**, 614–641 (1967).

The fluid RE model augments the resistive MHD equations.

$$\frac{\partial n_r}{\partial t} + \boldsymbol{\nabla} \cdot (n_r \boldsymbol{u_r}) = S_D(\mathcal{E}_{||}) + S_A(E_{||}) + D_r \nabla^2 n_r, \tag{1}$$

where n_r is the number density of runaways, S_D, S_A are sources, D_r is a numerical diffusion coefficient and

$$\mathcal{E}_{||} \equiv \frac{E_{||}}{E_D}, \quad \boldsymbol{u_r} = -c_r \hat{\mathbf{b}} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}, \quad c_r = \text{const.} \gg v_{th,e}, v_A$$

$$E_D = \frac{n_e e^3 \ln \Lambda}{4\pi\epsilon_0 T_e}.$$

And a modified Ohm's law:

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \eta(\boldsymbol{J} - \boldsymbol{J}_r) \tag{2}$$

This model is useful for disruption modeling.

- Cai and Fu use a fluid model in the M3D code to consider runaways effect on the resistive internal kink.[1]
- Bandaru, et al use a fluid model including sourcing in JOREK to simulate RE beam termination in JET experiments.[2]
- Liu, et al have used M3D-C1 calculations to self-consistently simulate resistive kinks in post-disruption plasmas.[3]

[1] H. Cai and G. Fu, Nuclear Fusion **55**, 22001 (2015).
[2] V. Bandaru et al., Plasma Physics and Controlled Fusion **63**, 10.1088/1361-6587/abdbcf (2021).
[3] C. Liu et al., Plasma Physics and Controlled Fusion **63**, 10.1088/1361-6587/ac2af8 (2021).

In this work we consider the linearized equations around a 0-β equilibrium supported by runaway electron current.

Lowercase letter variables are perturbed quantities, capital letter variables are equilibrium quantities:

$$
\begin{gathered}
\partial_t n_r + \boldsymbol{\nabla} \cdot (n_r \boldsymbol{U}_r + N_r \boldsymbol{u}_r) = 0, \;\; \boldsymbol{U}_r = -c_r \frac{\boldsymbol{B}}{B}, \;\; \boldsymbol{u}_r = -c_r \frac{\boldsymbol{b}_\perp}{B} \\
\rho \partial_t \boldsymbol{v} = \boldsymbol{j} \times \boldsymbol{B} + \boldsymbol{J} \times \boldsymbol{b}, \quad \partial_t \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{e} \\
\boldsymbol{\nabla} \times \boldsymbol{b} = \mu_0 \boldsymbol{j}, \\
\boldsymbol{e} = -\boldsymbol{v} \times \boldsymbol{B} + \eta (\boldsymbol{j} - \boldsymbol{j}_r) \\
\boldsymbol{j}_r = -e (n_r \boldsymbol{U}_r + N_r \boldsymbol{u}_{\boldsymbol{r}}) \\
\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0
\end{gathered}
$$

Uniform constant plasma density, ρ, and uniform, constant resistivity η. No equilibrium flow: $\boldsymbol{V} = 0$. $c_r > 0$ is the parallel speed of runaway electrons along magnetic field lines. Source terms and RE drift effects are neglected in this linear analysis.

Energetic particle beams in a background plasma are susceptible to long-wavelength instabilities.

The literature considers self-pinched beams of relativistic electrons moving through a resistive neutralizing background plasma. The fundamental instability picture is

- Perturbed particle orbits generate RE current.
- Eddy currents induced in the ohmic bulk plasma decay.
- The imperfect cancellation of the perturbed RE current by the ohmic plasma currents allows an overstable oscillation.

Some relevant key features of the analyses:

- Unstable modes are essentially transverse to the beam axis.
- The $m = 1$ is generally most unstable, and is referred to as the 'resistive hose' instability[4]
- The axial wavenumber, k is small.
- The perturbed motion of the background plasma is negligible (other than neutralizing the beam.)
- Growth rate is asymptotically linear in η in the small η regime

[4] M. N. Rosenbluth, Physics of Fluids **3**, 932–936 (1960), S. Weinberg, Journal of Mathematical Physics **5**, 1371–1386 (1964).

Instability requires only EM Field + RE current.

Low-frequency EM wave equation w/RE ($,\boldsymbol{j}_r,$) and Ohmic plasma currents $\sigma\boldsymbol{E}$:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} - \cancelto{\sim 0}{\omega^2 \boldsymbol{E}} = -4\pi i\omega \boldsymbol{j}_r - 4\pi i\omega\sigma\boldsymbol{E}$$

How $\boldsymbol{j}_r$ is calculated determines physics fidelity for RE orbits:

Rosenbluth uses relativistic kinetics:

$$\partial_t f_r + \frac{\boldsymbol{p}\cdot\nabla f}{\sqrt{1+p^2}} + \frac{e}{m}\left(\boldsymbol{E} - \frac{\boldsymbol{p}\times\boldsymbol{B}}{\sqrt{1+p^2}}\right)\cdot\nabla_{\boldsymbol{v}} f = 0,$$

$$\boldsymbol{j}_r = -\frac{e}{m}\int \frac{\boldsymbol{p} f}{\sqrt{1+p^2}} d^3p$$

Fluid model assumes $m = 0$
$p/m \to c_r$:

$$\boldsymbol{j}_r = \frac{e c_r n_r}{B}\boldsymbol{B},$$

$$e c_r \partial_t n_r + \nabla\cdot\boldsymbol{j}_r = 0.$$

RE's only move along $\boldsymbol{B}$

MHD flow and details of RE physics are non-essential for instability.

NIMROD[5] finds time-dependent solutions of the linearized MHD+RE equations

- NIMROD uses a Fourier representation of the axial direction in cylindrical geometry.
- Linear problems decouple harmonics: $f(r, \theta, z, t) \sim f(r, \theta, t)\exp(ikz)$, then time-evolution for each k from an arbitrary initial condition is computed.
- A high-order finite element representation is used in the r, θ plane.
- The equations are integrated in time with a semi-implicit scheme.

[5] C. Sovinec et al., J. Comp. Phys. **195**, 355 (2004).

The linearized MHD+RE system is solved with NIMROD, and a 1D radial eigenvalue code.

- NIMROD representation: $f(r,\theta,z,t) \sim f(r,\theta,t)\exp(ikz)$.
- Eigenvalue code representation: $f(r,\theta,z,t) \sim f(r)\exp(ikz + m\theta - i\omega t)$

NIMROD:

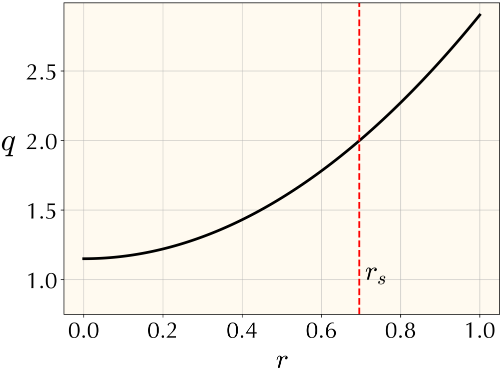
- 2D finite element representation in $(R,Z) \mapsto (r,\theta)$
- PDE IVP for $n_r, \boldsymbol{b}, \boldsymbol{v}$ given k
- Solution tends to the most unstable mode as $t \to \infty$

Eigenvalue Code:

- Assumes cylindrical symmetry
- 1D finite element representation in r
- ODE EVP for $n_r, \boldsymbol{b}, \boldsymbol{v}$ given k and m

Cylindrical equilibrium taken from Liu et. al[6] was used to benchmark implementation.

$$L = 20\pi, \quad a = 1,$$
$$B_z(r = 0) = 1, \quad \mu_0\rho = 1,$$
$$c_r = 20, \ \beta = 0,$$
$$q(r) = 1.15(1 + (1.234r)^2)$$

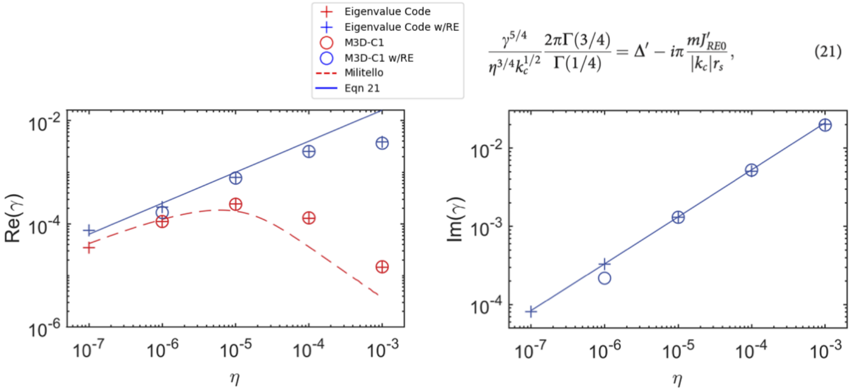


- All the equilibrium current is carried by runaways
- $q(0) > 1$
- In resistive MHD the most unstable mode is the $m = 2$ tearing mode

[6] C. Liu et al., Physics of Plasmas **27**, 10.1063/5.0018559 (2020).

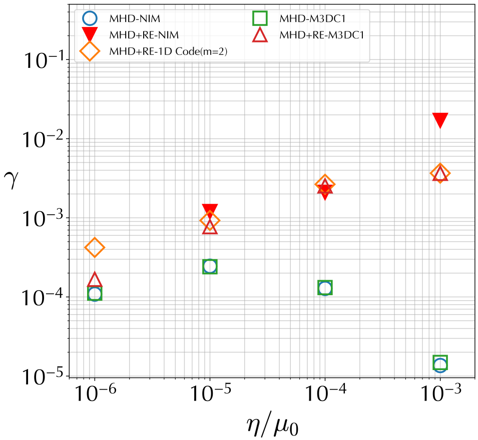
Results from Liu, et al. compare an analytic growth rate scaling with linear M3D-C1 calculations for the (2,1) tearing mode.[7]

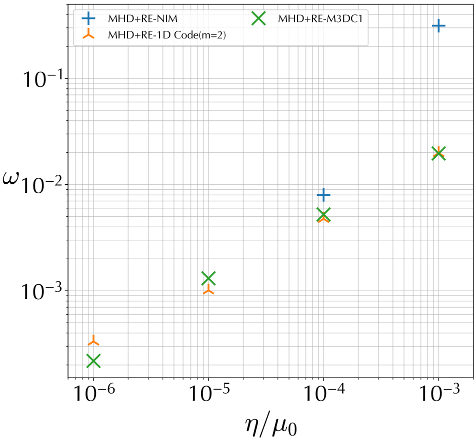
$$\frac{\gamma^{5/4}}{\eta^{3/4}k_c^{1/2}}\frac{2\pi\Gamma(3/4)}{\Gamma(1/4)} = \Delta' - i\pi\frac{mJ'_{RE0}}{|k_c|r_s}, \tag{21}$$



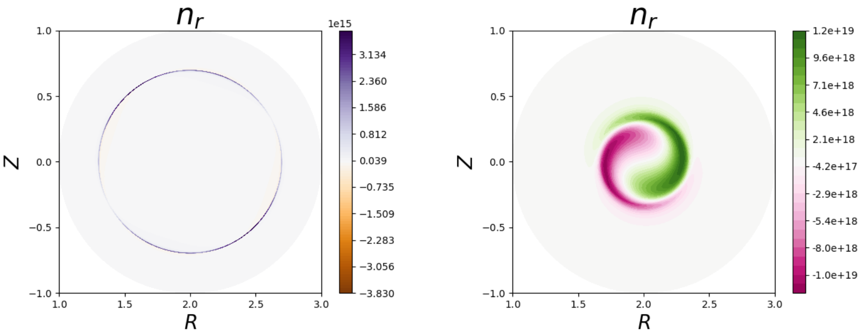
[7] C. Liu et al., Physics of Plasmas **27**, 10.1063/5.0018559 (2020).

Both NIMROD and eigenvalue code reproduces published result on tearing with REs for small η.





Radial structure of beam mode differentiates it from either the kink or tearing mode



The eigenfunction of the tearing mode (left) at $S = 10^4$ is localized near the $q = 2$ rational surface with poloidal mode number $m = 2$. The fastest growing mode at $S = 10^2$, has $m = 1$, and is not localized near the rational surface.

Neglecting velocity perturbations does not affect the mode.

Set $\boldsymbol{v} \sim 0$, and change variables to $\Lambda_r = \mu_0 e c_r N_r / B$, $\lambda_r = \mu_0 e c_r n_r / B$, then since $\partial_t \boldsymbol{B} = 0$:

$$\partial_t \lambda_r - \frac{c_r}{B} \left(\boldsymbol{B} \cdot \nabla \lambda_r + \boldsymbol{\nabla} \cdot (\Lambda_r \boldsymbol{b}_\perp) \right) = 0, \tag{3}$$

$$\partial_t \boldsymbol{b} = \frac{\eta}{\mu_0} \boldsymbol{\nabla} \times (\lambda_r \boldsymbol{B} + \Lambda_r \boldsymbol{b}_\perp) - \frac{\eta}{\mu_0} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}. \tag{4}$$

This simplified system was solved with the 1D eigenvalue code.

A further reduced model reveals that the instability is primarily driven by the gradient in the equilibrium runaway current.

Since the axial component $|b_z| \ll |b_r|, |b_\theta|$, simplify the system via $\boldsymbol{b} = \nabla\psi \times \hat{\boldsymbol{z}}$. Additionally, we neglect terms of order $r/R \sim B_\theta/B$, and assume $B_z \sim B_0 = \mathrm{const.}$ (large-aspect ratio).

$$iF(r) \equiv \frac{\boldsymbol{B}\cdot\nabla}{B},$$

$$(\omega + c_r F(r))\lambda_r = -c_r \frac{m\Lambda_r'}{r}\frac{\psi}{B},$$

$$\omega\psi - i\frac{\eta}{\mu_0}\left(\frac{1}{r}(r\psi')' - \frac{m^2}{r^2}\psi\right) = i\frac{\eta}{\mu_0}B_z\lambda_r.$$

The eigenvalues of this reduced system are also sought with a 1D spectral method.

This description of beam-plasma interaction is similar to resistive hose theory

The lowest order description of the beam problem given by Weinberg[8] is

$$\frac{1}{r}(rE'_{1z})' - \frac{m^2}{r^2}E_{1,z} + \frac{4\pi i\omega}{c^2}E_{1z} = -\frac{4\pi iev\omega}{c^2}n_1.$$

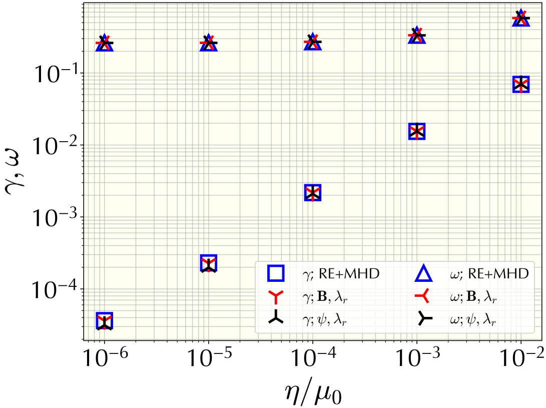
Where E_{1z} is the axial electric field, n_1 is the perturbed beam density, σ the bulk plasma conductivity, and v the axial beam velocity. This is equivalent to our equation for ψ after the substitutions $E_{1z} = i\omega\psi,\ 4\pi\omega\sigma/c^2 \to \mu_0/\eta,\ 4\pi/c^2(evn_r) \to B_z\lambda_r$:

$$\omega\psi - i\frac{\eta}{\mu_0}\left(\frac{1}{r}(r\psi')' - \frac{m^2}{r^2}\psi\right) = i\frac{\eta}{\mu_0}B_z\lambda_r.$$

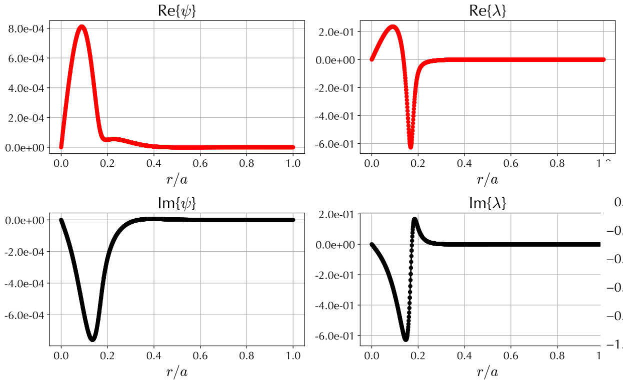
[8] S. Weinberg, Journal of Mathematical Physics **8**, 614–641 (1967).

The reduced models are sufficient to describe the instability.

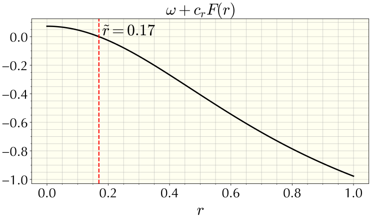
- EV calculations of growth rates and frequencies of the beam mode with $m = 1$ agree between all three models.
- The ability to neglect $\boldsymbol{v}$ suggests that the dimensionless number of importance is something other than the Lundquist number, S.(maybe $ac_r\mu_0/\eta$)



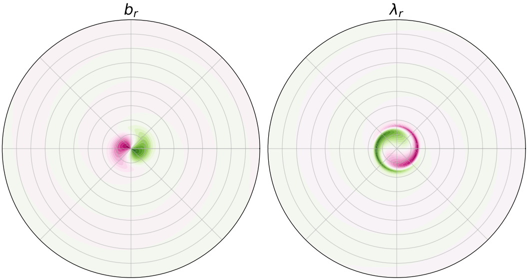
Radial profiles of reduced model eigenfunctions give some insight.

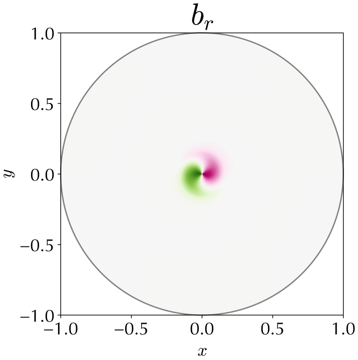


- The radial structure is localized within a region of width $\tilde{r}$ around the origin.
- The location of $\tilde{r}$ seems to be determined by a resonance of $\mathrm{Re}\{\omega\} + c_r F(\tilde{r}) = 0$.



Eigenvalue calculations from the 1D code observe the same mode structure and growth rates as in the NIMROD calculation.





Conclusions: Linear simulations have revealed an effect of RE current on resistive MHD instabilities.

- Prior analytic and computational results have shown that the linear behavior tearing and resistive kink modes are affected by the presence of runaway electron current.
- In a cylindrical $(2,1)$ tearing mode case, NIMROD and Eigenvalue calculations agree with published results for $\eta/\mu_0 \lesssim 10^4$.
- For $\eta/\mu_0 > 10^4$, there is a distinct, faster growing mode that is associated with the presence of the RE beam.
- The form of the reduced model suggests it is related to the resistive hose instability.

Acknowledgments

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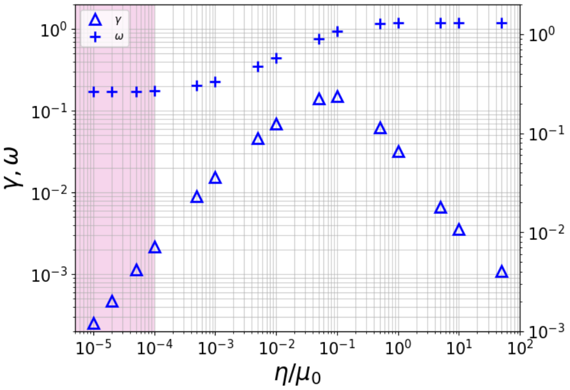
Special thanks to Dr. Liu for providing data for the tearing mode test case.

At large resistivity, eigenvalues scale with η the same way observed in the MHD simulation in the high-η regime.



- $\omega > \gamma$ for all η values for this mode
- In the tearing mode, $\omega \lesssim \gamma$ at low η.
- The shaded region indicates where the MHD tearing mode growth rates would be larger for this equilibrium.