Error field penetration and locking to the backward wave*

J. M. Finn$^1$, A. J. Cole$^2$ and D. P. Brennan$^3$

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Static error field interacting with finite frequency modes

- Known: the response to error fields is largest if the tearing mode is *weakly* stable.
- Observation: the response is largest if the Doppler-shifted tearing mode frequency is nearly at rest in the lab frame: \( \pm \omega_r + k\nu \approx 0 \)

\[
\tilde{\psi}(r_t) = -\frac{l_{21}}{\Delta' - \Delta(ik\nu)} \tilde{\psi}(r_w)
\]

- Real frequencies for tearing modes in several regimes: Glasser effect, diamagnetic propagation,...
- Torque (Maxwell stress) applied across the tearing layer:

\[
N_m = -\frac{k}{2} |\tilde{\psi}(r_t)|^2 \text{Im} \Delta(ik\nu)
\]

- Although \( |\tilde{\psi}(r_t)|^2 \) is maximum at \( \nu = \omega_r/k \), the torque is *zero* there. Weak driving torque \( \implies \) plasma locks to \( \nu \gtrsim \omega_r/k \), *not* to \( \nu \gtrsim 0 \).
Simplest layer response function $\Delta$ is the Viscoresistive (VR) tearing mode

$$\Delta(\gamma) = \frac{\mu^{1/6}}{\eta^{5/6}|k'||^{1/3}B^{1/3}} \gamma \quad (p' = 0) \quad \Delta(\gamma) = \gamma \tau_{vr}$$

Spontaneous modes have $\gamma \tau_{vr} = \Delta'$. Zero real frequency for both signs of $\Delta'$.

$$\Delta' = [\tilde{\psi}']_{rt}/\tilde{\psi}(r_t) \text{ for } \tilde{\psi}(r_w) = 0.$$

$$\tilde{\psi}(r_t) = -\frac{l_{21}}{\Delta' - \Delta(ik\nu)} \tilde{\psi}(r_w) = -\frac{l_{21}}{\Delta' - ik\nu \tau_{vr}} \tilde{\psi}(r_w)$$
Reconnected flux near maximum in locked torque balance state

The field $|\tilde{\psi}(r_t)|^2$ and torque $N_m$ vs. $\hat{v} = k\nu \tau$ for the VR regime have simple structure near $\hat{v} = 0$.
Equilibrium $\nu_0$ determined by flow drive (e.g. beams).

Steady state torque balance $N_v = N_m$ at tearing layer determines roots ($\hat{v}$), stable or unstable.

Viscous torque $N_v \sim \mu(\hat{v}_0 - \hat{v})$ intersects at 1 or 3 equilibria.
Locked state has $\hat{v} \geq 0$. 
Penetration is a bifurcation to a high reconnected flux, low flow state

Standard bifurcation picture gives penetration threshold in either error field amplitude or equilibrium flow.
Reconnected flux and torque depend on Doppler shift

\[ \text{Denom} = \Delta' - ikv \tau_{\text{vr}} = (\gamma - ikv) \tau_{\text{vr}}. \]  
\[ \gamma \text{ real } \implies \text{maximum of } |\tilde{\psi}(r_t)|^2 \text{ is at } \hat{\nu} \equiv kv \tau_{\text{vr}} = 0 \]

Locus of roots for VR tearing mode, \( \omega \to \omega + kv \)

Max. linear response is for \( \Delta' \lesssim 0 \) and \( \nu = 0 \).

\[ N_m = -\frac{k^2 l_{21}^2 |\tilde{\psi}(r_w)|^2}{2} \frac{v \tau_{\text{vr}}}{\Delta'^2 + k^2 v^2 \tau_{\text{vr}}^2} \propto -\frac{v}{c_0^2 + v^2} \] (Well quoted)
Resistive-inertial (RI) regime, $\rho' = 0$

$$\Delta(\gamma) = \frac{\rho^{1/4}}{\eta^{3/4}|k'||^{1/2}B^{1/2}} \gamma^{5/4} \quad \Delta(\gamma) = \gamma^{5/4} \tau_{ri}^{5/4}$$

Real $\gamma$ for $\Delta' > 0$; For $\Delta' < 0$, $\gamma \tau_{ri} = |\Delta'|^{4/5} e^{\pm 4\pi i/5}$

$$|\tilde{\psi}(r_t)|^2 = \frac{l_{21}^2 |\tilde{\psi}(r_w)|^2}{|\Delta' - (ikv \tau_{ri})^{5/4}|^2} \sim \frac{1}{|(\gamma \tau_{ri})^{5/4} - (ikv \tau_{ri})^{5/4}|^2} \sim \frac{1}{|\gamma - ikv|^2}$$

N. B. denom is $(\Delta' - \Delta_r(ikv \tau_{ri}))^2 + \Delta_i(ikv \tau_{ri})^2$, not $\Delta'^2 + (\cdots)^2$

Roots for RI mode,

$$\omega \to \omega + kv.$$  

Off-axis peaks but $\mathcal{N}_m(\nu)$ qualitatively the same.
RI with $p'$ and parallel dynamics leads to “Glasser effect” and complex roots

$$
\Delta' = \Delta(\gamma \tau) \equiv (\gamma \tau)^{5/4} - \frac{\pi D}{4(\gamma \tau)^{1/4}} \ldots D \sim -p'(1 - q^2) < 0
$$

$\Delta(\gamma \tau)$ vs. $\gamma \tau$ showing $\Delta_{\text{min}}$

Locus of roots for RI $D < 0$, $\omega \rightarrow \omega + kv$.

C.c. complex roots if $\Delta' < \Delta_{\text{min}} = 2.0|D|^{5/6}$, These roots stabilized if $\Delta' < \Delta_{\text{crit}} = 1.13|D|^{5/6}$
RI with Glasser effect has penetrated state near zero torque at significant $\hat{\nu}$

$$N_m \propto -\frac{\Delta_i(ik\nu\tau_{ri})}{(\Delta' - \Delta_r(ik\nu\tau_{ri}))^2 + \Delta_i(ik\nu\tau_{ri})^2}$$

Numerator $= 0$ near where denominator minimum; $\Delta_i = 0$ at $\omega_r = k\nu$

- Note pronounced peaks in $|\tilde{\psi}(r_t)|^2$ off axis.
- Viscous torque $N_v(\nu) \propto \mu(\nu_0 - \nu)$ for small $\mu$ intersects at $\nu \gtrsim \omega_r/k$.
  Fields are locked to the static error field, but the plasma flow is locked to finite value, $\nu \gtrsim \omega_r/k$ rather than $\nu \gtrsim 0$.
- For very small $\mu$ two other states are possible, L-stable; R-unstable.
Finite $\hat{\nu}$ persists in RI with Glasser as $\Delta' \rightarrow \Delta_c$—
Numerator $= 0$ near where denominator minimum; $\Delta_i = 0$ at $\omega_r = k\nu$

$|\tilde{\psi}(r_t)|^2$ for $\Delta'$ strongly stable

$|\tilde{\psi}(r_t)|^2$ for $\Delta'$ marginally stable
Penetration bifurcation diagram in RI

Plasma velocity locks to $v \gtrsim \omega_r/k$ and penetrated flux has maximum for $\hat{v}_0 > 0$.
VR regime with $\rho'$ and parallel dynamics also has complex roots

Numerical computations of the VR dispersion relation, using the constant-$\psi$ approximation

$\Delta(\hat{v})$ is non-monotonic in a range of $c_s$. $\Delta' \lesssim \Delta_2$ or $\Delta' \gtrsim \Delta_1 \implies$ complex roots. Glasser effect in VR!
Glasser Effect in VR Regime

Locus of roots similar to RI. Torque curve similar too.

Two new roots for negative $\nu$ present in VR too.
Nonlinear effects

- For $|\tilde{\psi}(r_w)|$ large enough and $|\gamma - ikv|$ small and $\Delta' \lesssim 0 |\tilde{\psi}(r_t)|$ and hence locked island can become large enough $W \sim \delta$ to enter the Rutherford regime.

- For $|\tilde{\psi}(r_t)|$ larger, the *Scott regime* can be entered, when $k' W c_s \sim \omega_r$. Island flattening (for $\omega_r = \omega_\ast$).

![Graph showing nonlinear effects](image-url)
Conclusions

- For tearing modes with real frequencies, the peak response $\tilde{\psi}(r_t)$ is for weakly stable modes but also
- The peak response is for $\nu = \omega_r/k$ for the backward wave.
- Torque $N_m$ is zero at $\nu = \omega_r/k$; for small driving torque, the fields lock to zero frequency, but the plasma locks to $\nu \gtrsim \omega_r/k$ rather than $\nu \gtrsim 0$. Source of flow.
- Effect seen for RI with $D < 0$ and parallel dynamics; Glasser effect and so $\nu = \omega_r/k$ in VR! There are two new flow roots, L stable, R unstable.
- Who cares? E.g. error fields with $(m_1, n_1)$ and $(m_2, n_2)$ w/ $m_1/n_1 \neq m_2/n_2$; plasma can lock to two different velocities at $q(r) = m_1/n_1$ and $q(r) = m_2/n_2$ and with $NTV$ can lead to smooth rotation shear in between.
Conclusions

- In other regimes, there are real frequencies, often $\omega_r \propto \omega_*$, not c.c.
- A point about regimes: the unlocked (high-slip) state can be in one tearing regime and the locked state in another (or even be nonlinear).
- For slow flow, e.g. $v/v_A \sim 10^{-3}$ and very stable, locked state has small $\tilde{\psi}(r_w)$ and small islands, $W \sim \delta$ – Rutherford regime. These calculations with $v \gtrsim \omega_r/k$ are still qualitatively OK.
- For locked state with $\tilde{\psi}(r_w)$ even larger (larger flow or larger $\Delta'$), Scott regime $k' \parallel Wc_s \sim \omega_r$ ... pressure gradient flattens due to sound wave and propagation slows.
- For $\omega_r$ due to pressure-curvature, does a Scott-effect occur? Probably.