Helical Temperature Perturbations Associated with Radially Asymmetric Magnetic Island Chains in Tokamak Plasmas

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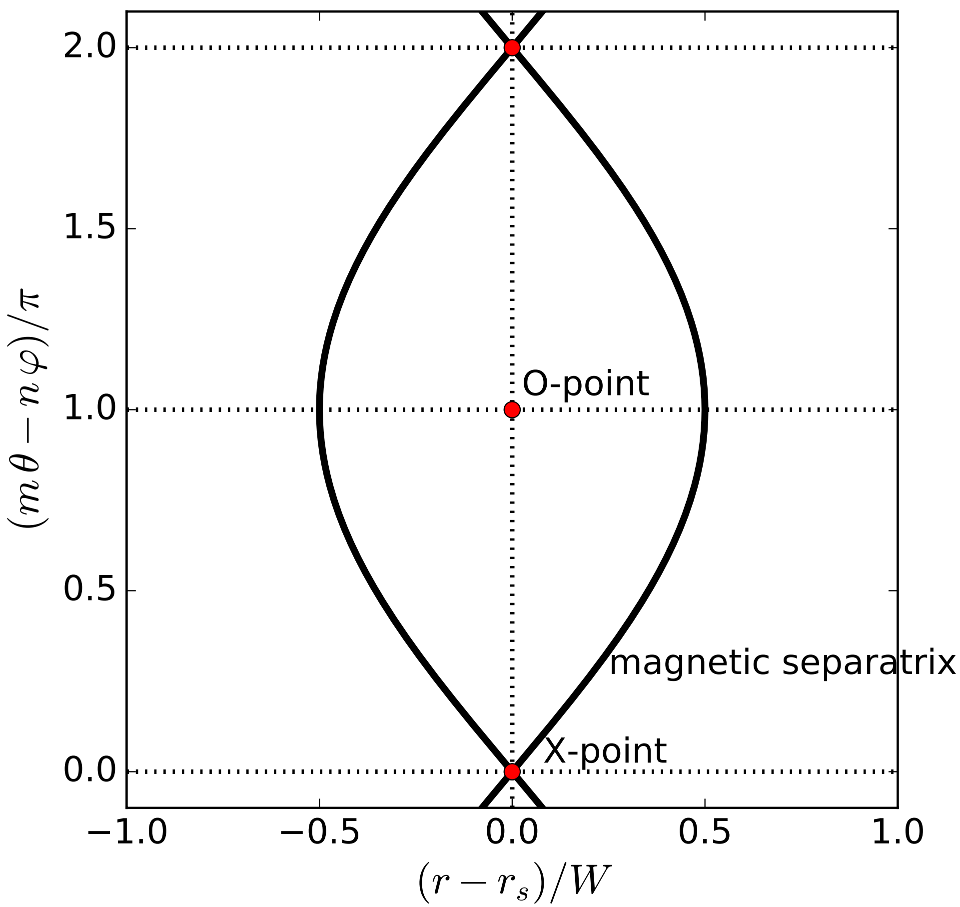
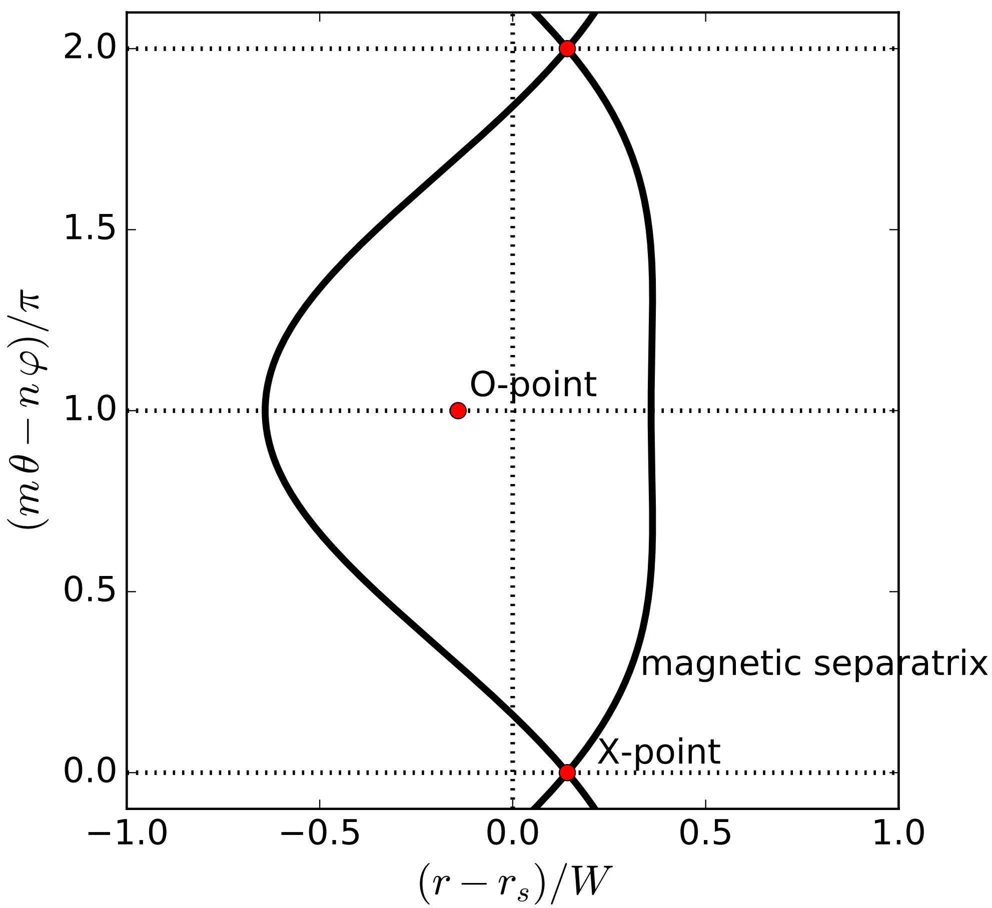
Conventional “Rutherford” Magnetic Island



Local helical flux: $\psi(r, \theta, \varphi) = \psi_0(r) + \Psi\cos(m\,\theta - n\,\varphi)$.

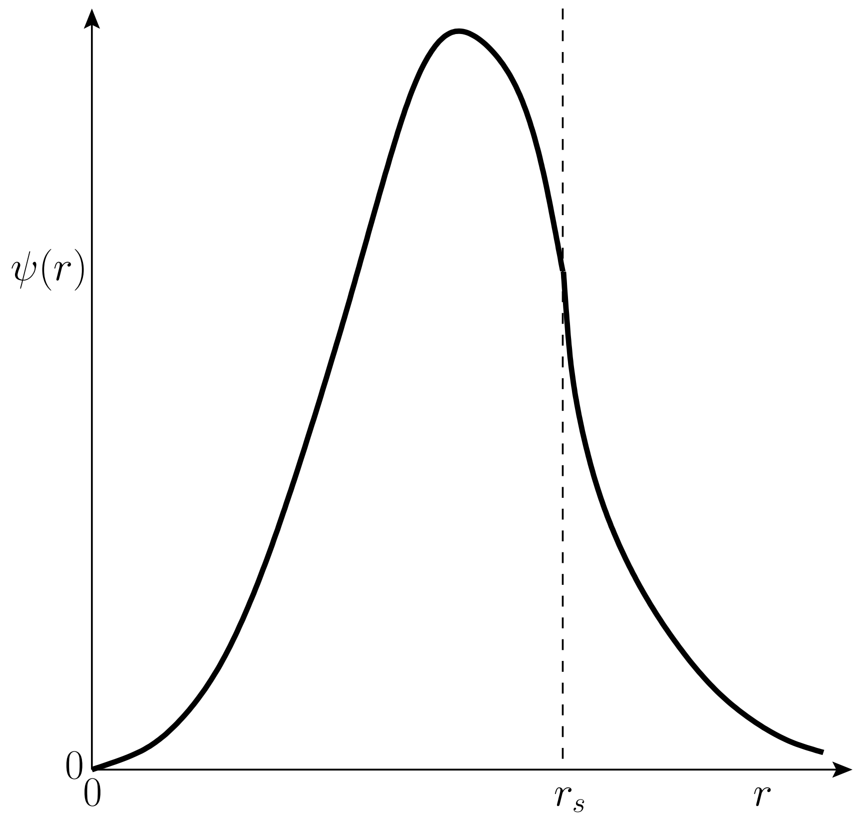
Realistic Magnetic Island



Local helical flux: $\psi(r,\theta,\varphi)=\psi_0(r)+\Psi\,[1-\delta\,(r-r_s)]\cos(m\,\theta-n\,\varphi)$.

Realistic Tearing Eigenfunction

$$\psi(r,\theta,\varphi) = \psi_0(r) + \psi(r)\,\mathrm{e}^{\mathrm{i}\,(m\,\theta - n\,\varphi)}$$



Distortion of magnetic island emanates from strong average negative gradient of eigenfunction at rational surface.

Effect of Radial Asymmetry on Island Stability

- White, Gates, and Brennan [a] find that asymmetry gives rise to destabilizing term in Rutherford island width evolution equation whose magnitude is proportional to island width. Term plays pivotal role in new theory of density limit disruptions.
- Hastie, Militello, and Porcelli [b] find similar destabilizing term in *thin-island limit* (when island too thin to flatten temperature profile), but find no such term in opposite *thick-island limit*.
- Arguments of White, Gates, and Brennan seem more applicable to thick-island than thin-island limit??
- Aim of talk is to resolve disagreement by extending standard Rutherford theory to take radial asymmetry into account.

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[a] Phys. Plasmas **22**, 022514 (2015).
[b] Phys. Rev. Lett. **95**, 065001 (2005).

Fundamental Model

- Nonlinear resistive-MHD analysis of magnetic island evolution. Incorporates non-inductive bootstrap current, plus asymmetric heat transport parallel and perpendicular to magnetic field-lines.
- Assumes radially thin island. Assumes uniform parallel inductive electric field in island region, small uniform temperature gradient in absence of island chain, and $\eta \propto T^{-3/2}$.
- Assumes that perturbed current density in island region small compared to background density, and neglects plasma inertia in vorticity equation. Follows that current density is magnetic flux-surface function.

Fundamental Definitions

- Normalized coordinates: $X = (r - r_s)/w$, $\zeta = m\,\theta - n\,\varphi$. Here, $W = 4\,w$ is full island width.
- Normalized helical flux: ψ. Normalized perturbed temperature: δT. Normalized perturbed current density: $\delta J = \delta J(\psi)$.
- Equilibrium magnetic shear: s_s. Equilibrium magnetic shear-length: L_s. Equilibrium temperature gradient scale-length: L_T. Critical island width (above which δT flattened): w_c.
- Resistive evolution time-scale: τ_R. Bootstrap parameter: $\alpha_b = f_s\,\beta_p$. Tearing stability index: Δ'.
- Poisson bracket: $[A, B] \equiv \partial_X A\,\partial_\zeta B - \partial_X B\,\partial_\zeta A$.
- Flux-surface average operator: $\langle [A, \psi] \rangle \equiv 0$, for general A.

Fundamental Equations

$$\partial_X^2 \psi = 1,$$

$$0 = \left(\frac{w}{w_c}\right)^4 [[\delta T, \psi], \psi] + \partial_X^2 \delta T,$$

$$\delta T(X, \zeta)|_{\lim |X| \to \infty} = X,$$

$$\delta J(\psi) = -\frac{\tau_R}{\langle 1 \rangle} \frac{\partial}{\partial t} \left[\langle \left(\frac{w}{r_s} \right)^2 \psi \rangle \right] + \alpha_b \frac{L_s}{L_T} \left(\frac{\langle \partial_X \delta T \rangle}{\langle 1 \rangle} - 1 \right)$$
$$- \frac{3}{2} \frac{w}{L_T} \left(\frac{2}{s_s} - 1 \right) \frac{\langle \delta T \rangle}{\langle 1 \rangle},$$

$$\Delta' w = -2 \int_{-\infty}^{\infty} \oint \delta J \cos\zeta \, dX \frac{d\zeta}{2\pi}.$$

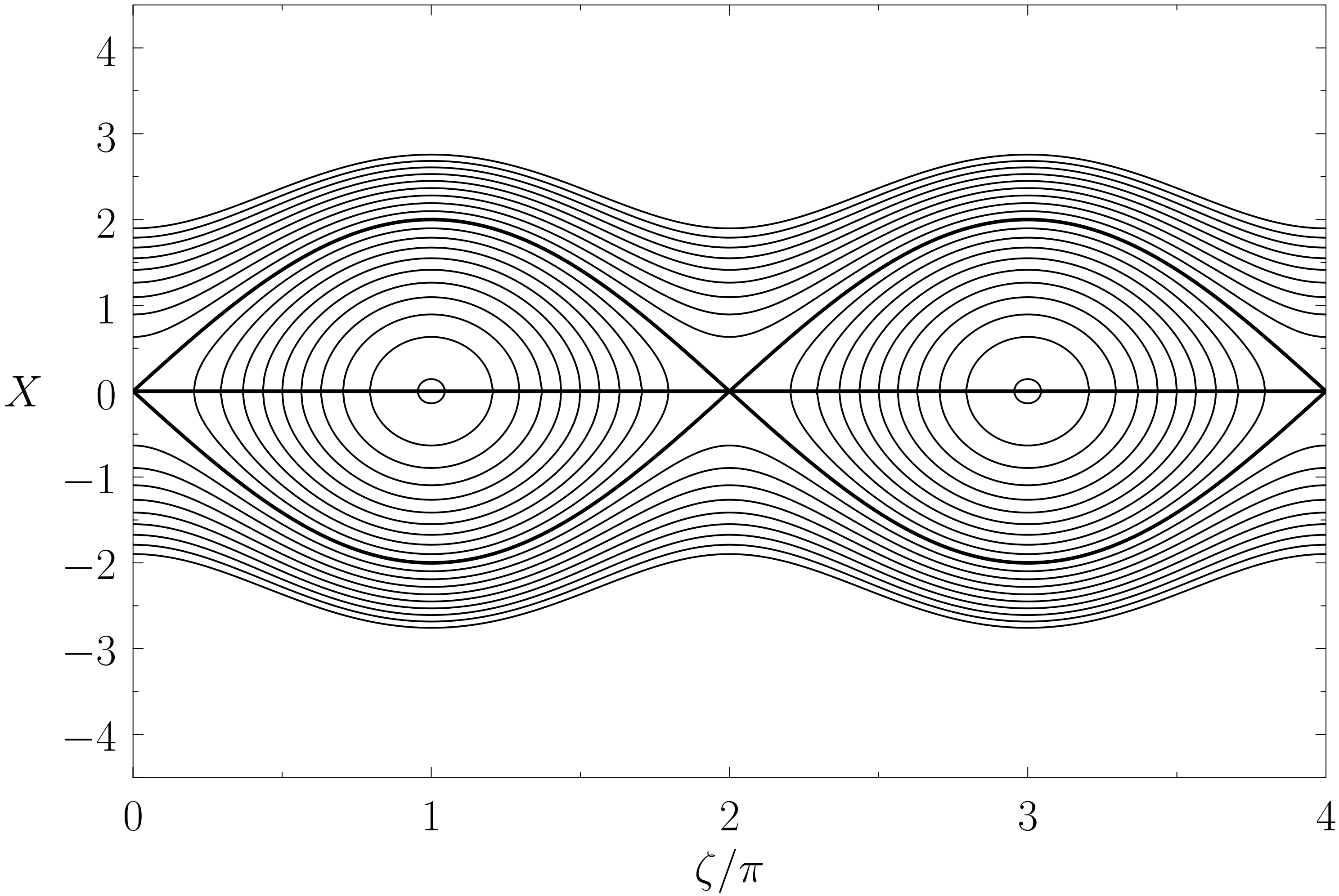
Model Magnetic Flux Surfaces

$$\psi(X,\zeta) = \Omega(X,\zeta),$$

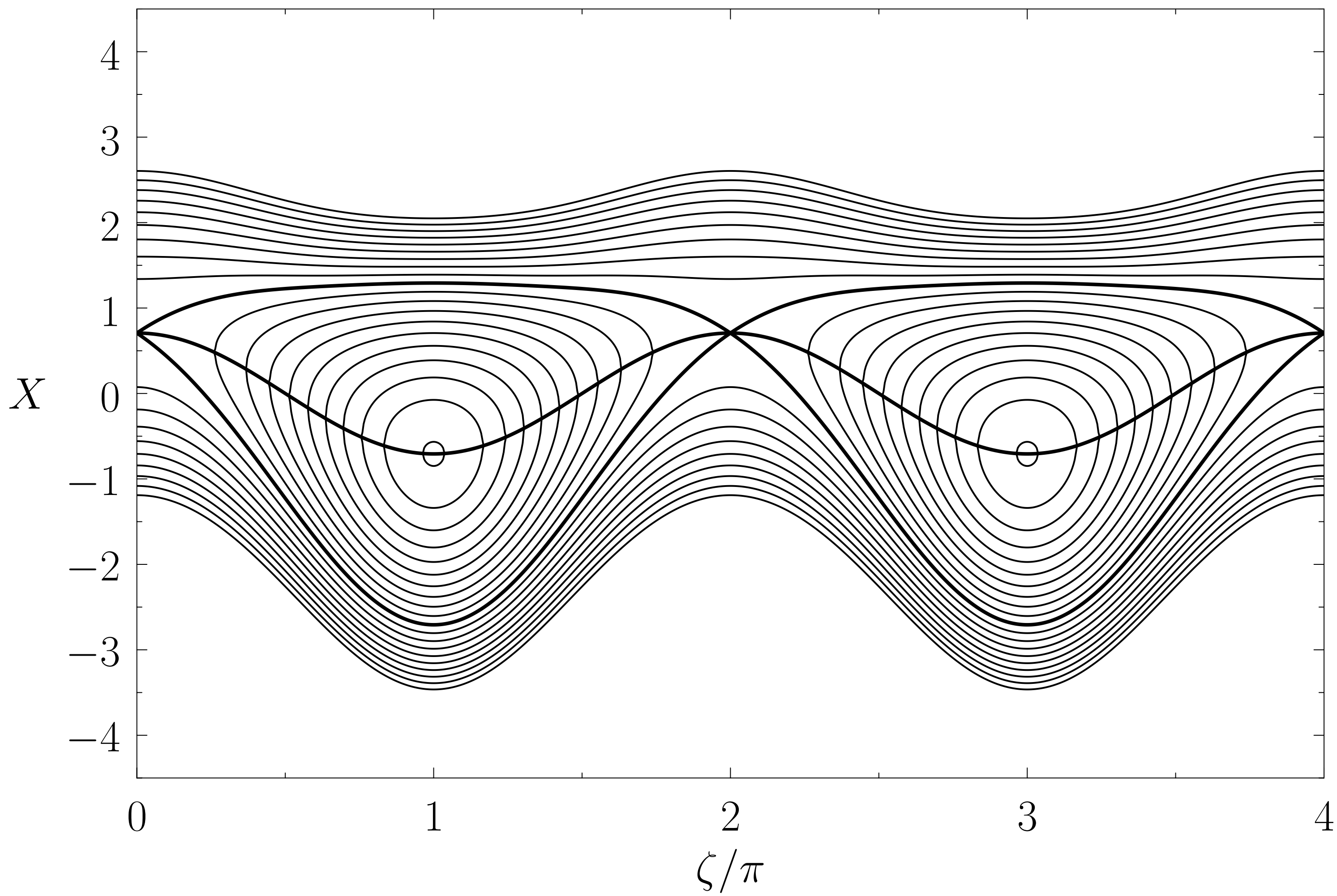
$$\Omega(X,\zeta) \equiv \frac{1}{2}X^2 + \cos(\zeta - \delta^2 \sin\zeta) - \sqrt{2}\,\delta\,X\cos\zeta + \delta^2\cos^2\zeta.$$

- Solution of $\partial_X^2\psi = 1$.
- δ, where $0 < \delta < 1$, controls radial asymmetry.
- Magnetic separatrix at $\Omega = 1$, and full island width $4\,w$, irrespective of value of δ.

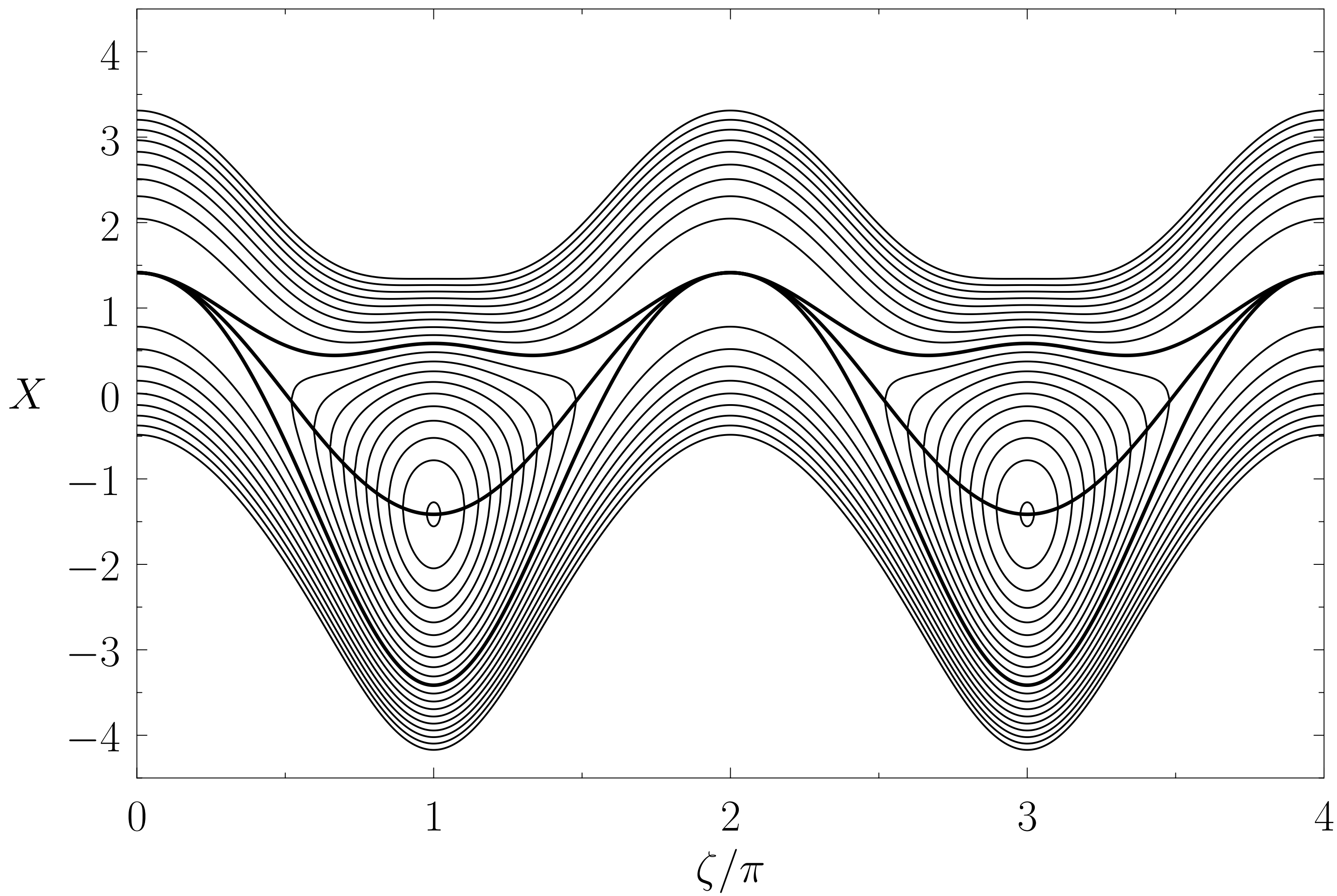
Ω-Contours in X-ζ Space: $\delta = 0$



Ω-Contours in X-ζ Space: $\delta = 0.5$



Ω-Contours in X-ζ Space: $\delta = 1.0$



Coordinate Transformation

$$Y = X - \sqrt{2}\,\delta\,\cos\zeta,$$

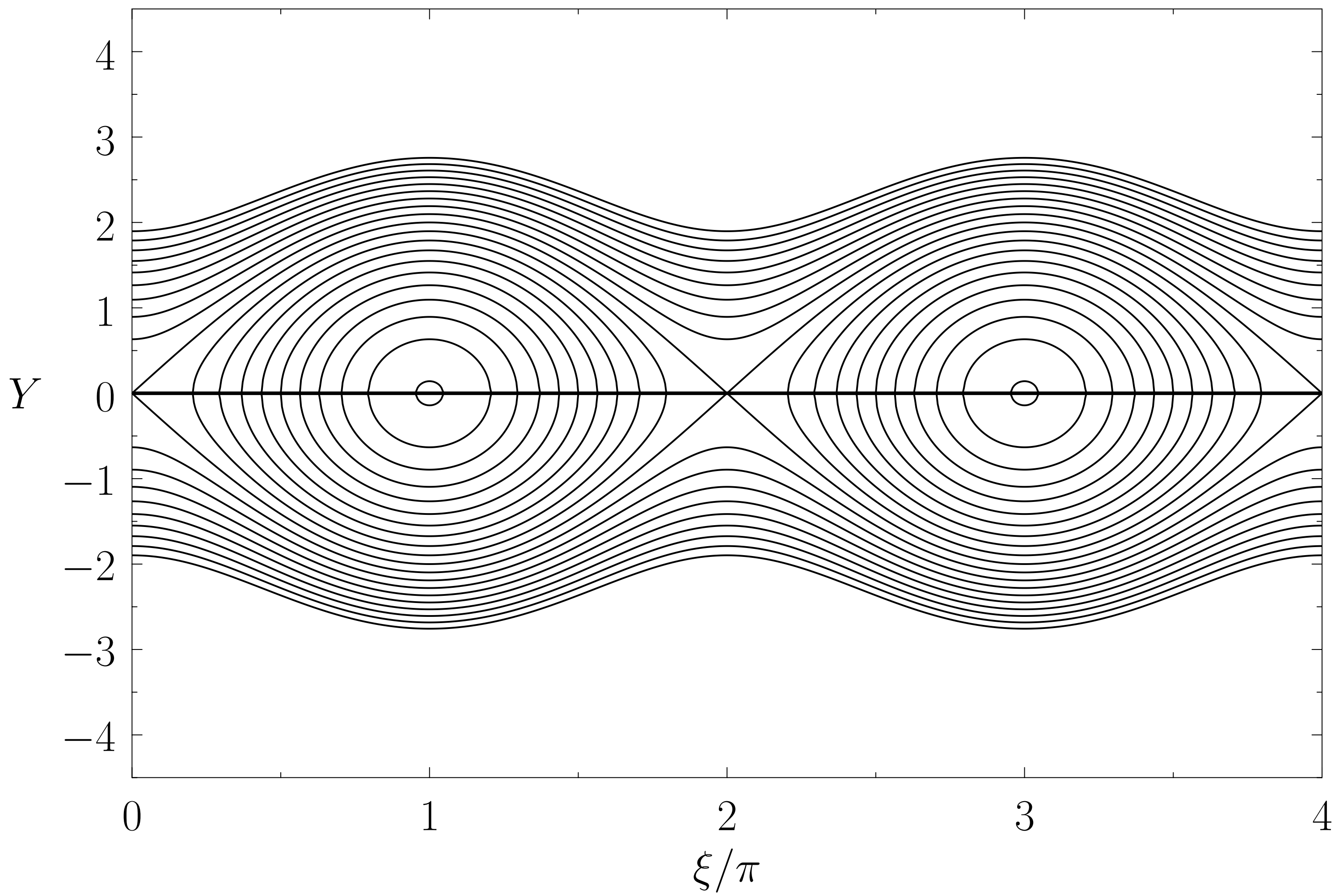
$$\xi = \zeta - \delta^{2}\,\sin\zeta.$$

- Flux surfaces map to

$$\Omega(Y, \xi) = \frac{1}{2}Y^{2} + \cos\xi,$$

irrespective of value of δ.

Ω-Contours in Y-ξ Space: Arbitrary δ



Flux-Surface Average Operator

$$\langle A \rangle = \int_{\xi_0}^{2\pi-\xi_0} \frac{\sigma(\xi)\, A_+(\Omega,\xi)}{\sqrt{2\,(\Omega - \cos\xi)}}\,\frac{d\xi}{2\pi} \qquad \text{for } -1 \leq \Omega \leq 1,$$

$$\langle A \rangle = \int_{0}^{2\pi} \frac{\sigma(\xi)\, A(\Omega,\xi)}{\sqrt{2\,(\Omega - \cos\xi)}}\,\frac{d\xi}{2\pi} \qquad \text{for } \Omega > 1,$$

where $\xi_0 = \cos^{-1}(\Omega)$, and

$$\sigma = 1 + 2 \sum_{n=1,\infty} J_n(n\,\delta^2)\,\cos(n\,\xi),$$

and

$$A_\pm(Y,\zeta) = (1/2)\left[A(Y,\zeta) \pm A(-Y,\zeta)\right].$$

Rutherford Equation

$$G_1\,\tau_R\,\frac{d}{dt}\left(\frac{W}{r_s}\right) = \Delta'\,r_s + G_2\,\alpha_b\,\frac{L_s}{L_T}\,\frac{r_s}{W} + G_3\,\frac{r_s}{L_T}\left(\frac{2}{s_s} - 1\right),$$

where

$$G_1 = 2\int_{-1}^{\infty}\frac{(\langle\cos\xi\rangle + \delta^{\,2}\,\langle\sin\xi\,\sin\zeta\rangle)\,\langle\cos\zeta\rangle}{\langle 1\rangle}\,d\Omega,$$

$$G_2 = 16\int_{-1}^{\infty}\frac{\langle\partial_Y\delta T_-\rangle\,\langle\cos\zeta\rangle}{\langle 1\rangle}\,d\Omega,$$

$$G_3 = -6\int_{-1}^{\infty}\frac{\langle\delta T_+\rangle\,\langle\cos\zeta\rangle}{\langle 1\rangle}\,d\Omega.$$

Narrow-Island Limit: $w \ll w_c$

$$\delta T_+(Y, \zeta) = 0,$$

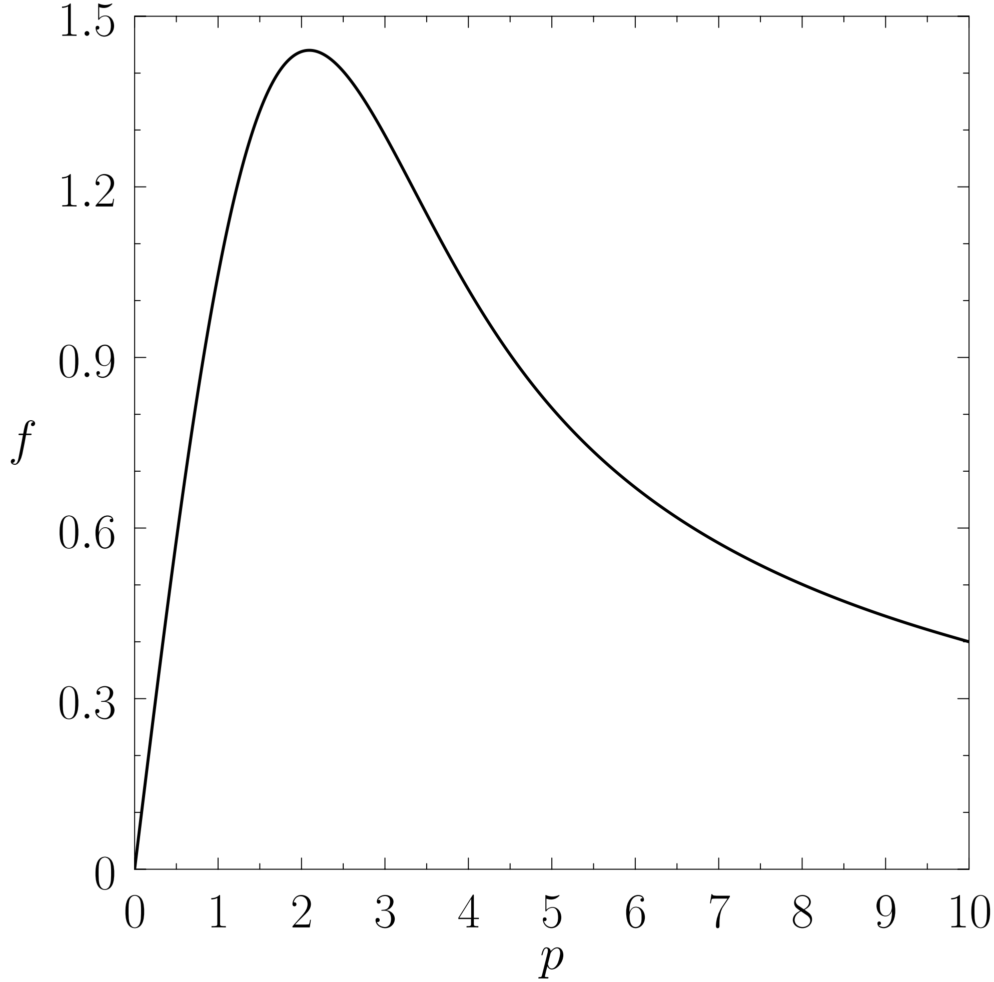
$$\delta T_-(Y, \zeta) = Y + \sum_{n=1,\infty} \frac{\sqrt{2\,n}}{4}\,\frac{w}{w_c}\left[(-1)^{n-1}\,J_{n-1}(\delta^{\,2}) + J_{n+1}(\delta^{\,2})\right]$$
$$\times\, f\left(\sqrt{2\,n}\,\frac{w}{w_c}\,Y\right)\cos(n\,\zeta),$$

where $f(p)$ is the solution of

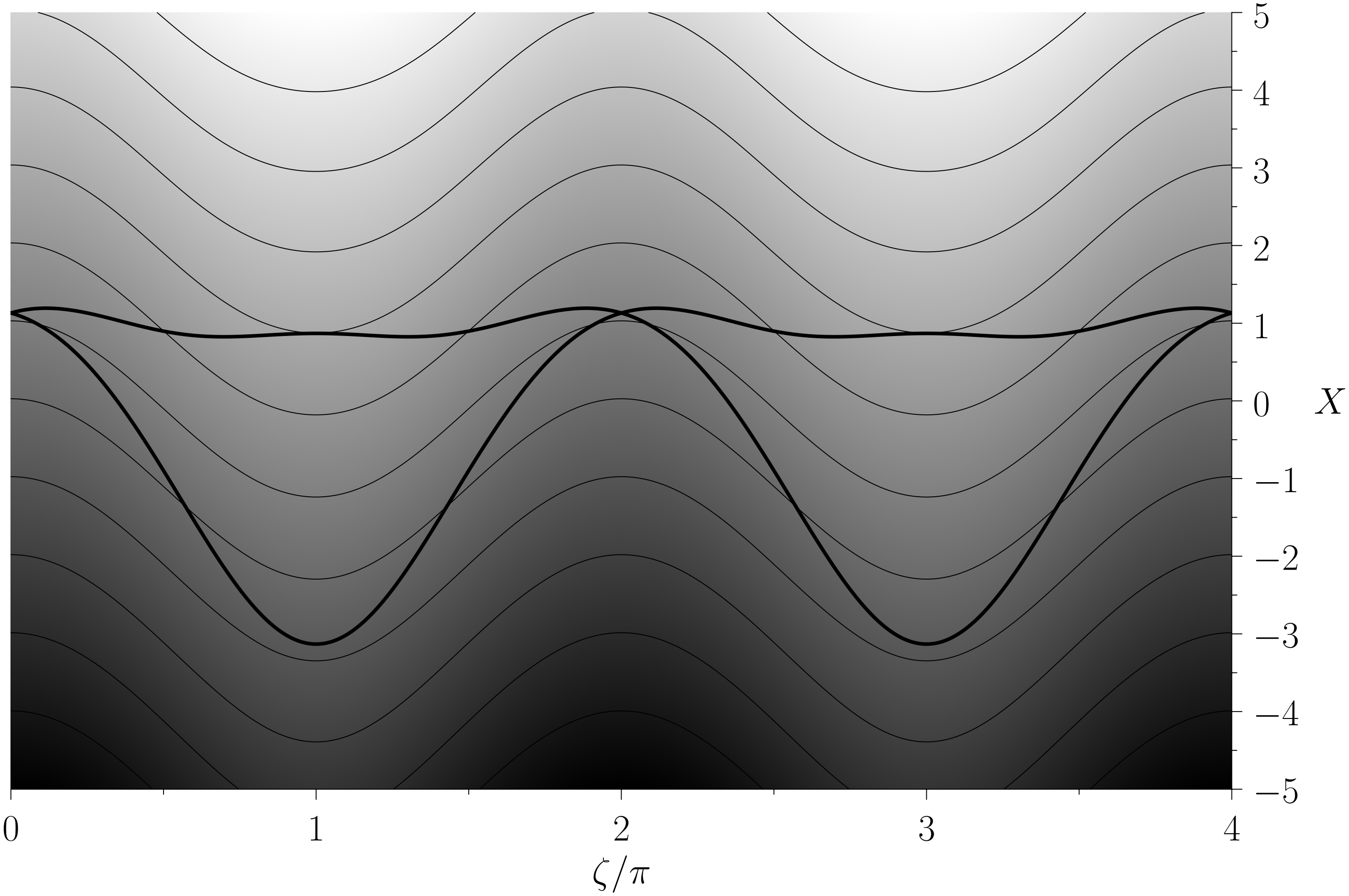
$$d^2 f/dp^{\,2} - p^{\,2}\, f/4 = -p$$

that satisfies $f(0) = 0$, and $f \to 0$ as $|p| \to \infty$.

f(p)



Contours of δT in X-ζ Space: $\delta = 0.8$



Narrow-Island Limit: Effect of Radial Asymmetry

- Because $\delta T_+ = 0$, follows that $B_3 = 0$ identically. No destabilization term due to island asymmetry in narrow-island limit.

- Destabilization term found by Hastie, Militello, and Porcelli [a] spurious, because authors incorrectly assumed $\eta = \eta(X)$ in narrow-island limit.

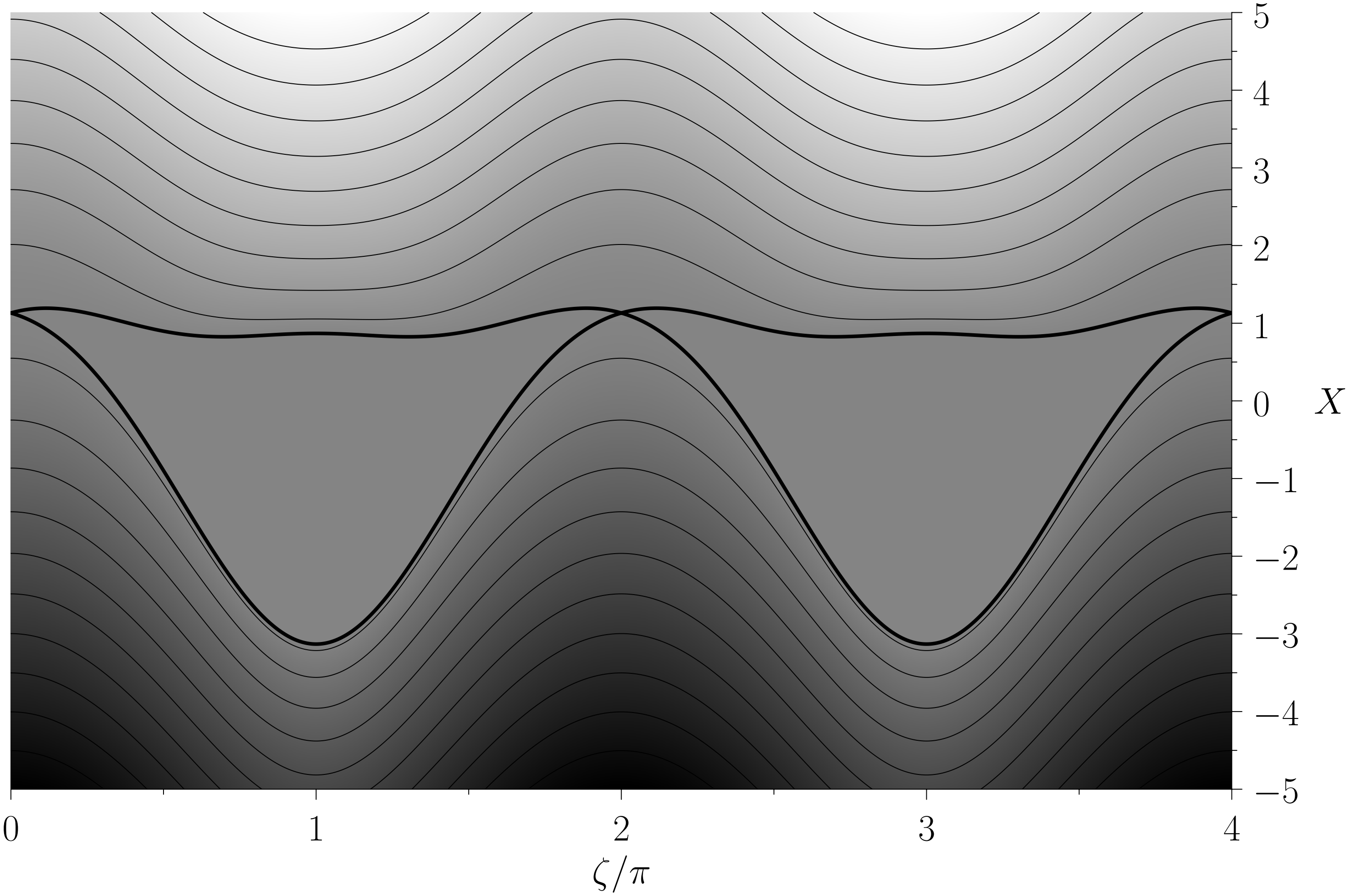
[a] Phys. Rev. Lett. **95**, 065001 (2005).

Wide-Island Limit: $w \gg w_c$

$$\delta T_+(\Omega) = 0,$$

$$\delta T_-(\Omega) = \begin{cases} 0 & -1 \leq \Omega \leq 1 \\ \int_1^{\Omega} \frac{d\Omega'}{\langle Y^2\rangle(\Omega')} & \Omega > 1 \end{cases}.$$

Contours of δT in X-ζ Space: $\delta = 0.8$



Wide-Island Limit: Effect of Radial Asymmetry

- Because $\delta T_+ = 0$, follows that $B_3 = 0$ identically. No destabilization term due to island asymmetry in wide-island limit.
- Destabilization term found by White, Gates, and Brennan [a] probably spurious, because calculation only heuristic in nature.

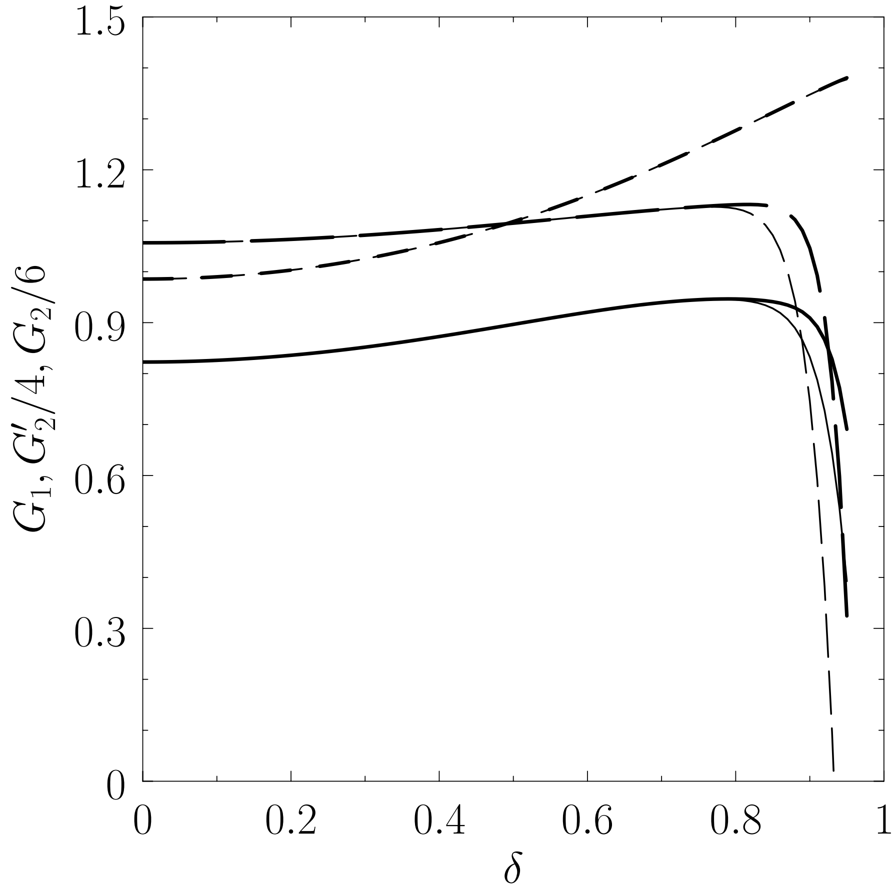
[a] Phys. Plasmas **22**, 022514 (2015).

Generalized Rutherford Equation

$$G_1\,\tau_R\,\frac{d}{dt}\left(\frac{W}{r_s}\right) = \Delta'\,r_s + \alpha_b\,\frac{L_s}{L_T}\,\frac{G_2\,G_2'\,W\,r_s}{G_2'\,W^2 + G_2\,W_c^2},$$

where $G_1 = G_1(\delta)$, etc., and $W_c = 4\,w_c$.

Constants in GRE



Solid, short-dashed, and long-dashed curves show G_1, $G_2'/4$, and $G_2/6$. Thick curves calculated with 15 harmonics. Thin curves calculated with 10 harmonics.

Summary

- Radial asymmetry has surprisingly little effect on evolution of Rutherford island.
- In particular, there is no evidence that asymmetry leads to new destabilizing term in Rutherford equation that scales as W.
- Physical Interpretation: An asymmetric island is a symmetric island plus a uniform (twisting parity) kink of same helicity. No reason to suppose that addition of such a kink would radically modify island stability. (Anymore than addition of a perturbation of different helicity would be expected to radically modify island stability.)