Generation of Seed Runaway Electrons: Spatio-Temporal Effects in Dynamic Scenarios

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IAEA/PPPL Theory and Simulation of Disruptions Workshop

Virtual meeting

July 19-23, 2021

A BRIEFF SUMMARY / ADVERTISEMENT OF RECENT WORK AT ORNL

- Seed runaway electrons problem
- Dissipation of runaway electrons in post-disruption plasmas
- Polarization of synchrotron emission

THE SEED RUNAWAY ELECTRON PROBLEM

- For a given plasma state (temperature drop, electric field evolution, magnetic field stochasticity, etc...), how many electrons become runaways before the second generation (avalanche) kicks in?
- Not knowing this is one of the weakest links in the assessment of the potential dangers of runaways in ITER and beyond
- The exponential growth predicted/assumed in the avalanche second generation process depends critical on the seed density
- The seed production depends on not well understood process including the nontrivial spatiotemporal evolution of the magnetic field stochasticity and the plasma cooling history
- This problem is one of the main deliverables of the DOE Theory Performance Targets for the SCREAM SciDAC project

THE SEED RUNAWAY ELECTRON PROBLEM

- What is needed is an accurate computational tool that allows the efficient/fast exploration of different disruption scenarios
- At the minimum, this tool should be able to incorporate arbitrary time dependences in model parameters as well as spatial effects (loss of confinement)
- Here we present recent progress on the ongoing development of a unique computational approach to this problem
- The first installment of this idea was published some time ago [Zhang and del-Castillo-Negrete, Phys. Plasmas 24, 092511 (2017).]
- Recently, we have made significant progress
    - M.Yang, G. Zhang, D. del-Castillo-Negrete, and M.Stoyanov, Accepted in Journal of Computational Physics (2021). https://arxiv.org/abs/2104.14561
    - D. del-Castillo-Negrete, M.Yang, M. Beidler G. Zhang, IAEA. 28th Int. Conference. IAEA-CN-286/101. Online (2021).

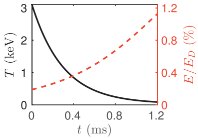
STANDARD APPROACH FOR THE COMPUTATION OF SEED RE DENSITY

- Solve the FP equation for a Maxwellian i.c. to get $f(\mathbf{r},\mathbf{p},t)$
- Prescribe the "runaway region", Ω_{RE}, based on a model and/or physical intuition
- Integrate $f(\mathbf{r},\mathbf{p},t)$ over the runaway region

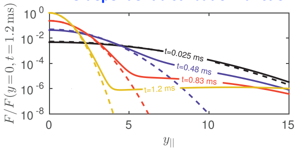
$$n_{RE}(t) = \int_{\Omega_{RE}} f(\mathbf{r},\mathbf{p},t)\, d\Omega$$

Example (among several others in the literature):

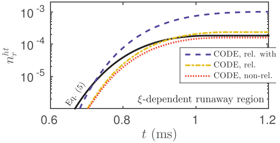
Cooling model and E field



Time-dependent distribution function



Seed runaway electron density

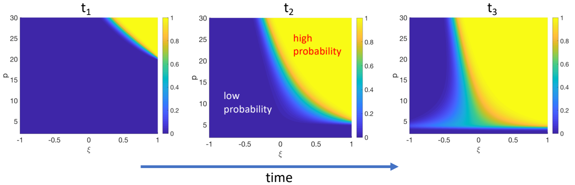


A. Stahl, et al., Nucl. Fusion **56** (2016) 112009

PROBABILISTIC APPROACH

- Compute the probability, P_{RE}, that an electron located at $(\mathbf{r}, \mathbf{p})$ will runaway at or before time t

Probability or runaway in (cos (pitch), momentum) space as function of time



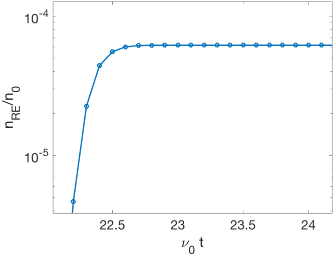
- Integrate over the whole space

RE seed density

$$n_{RE}(t) = \int P_{RE}(\mathbf{r}, \mathbf{p}, t)\, f_0(\mathbf{r}, \mathbf{p})\, d\Omega$$

Probability of runaway

Initial condition e.g., Maxwellian



ADVANTAGES OF THE PROBABILISTIC APPROACH: P_{RE} IS INDEPENDENT OF THE INITIAL CONDITION

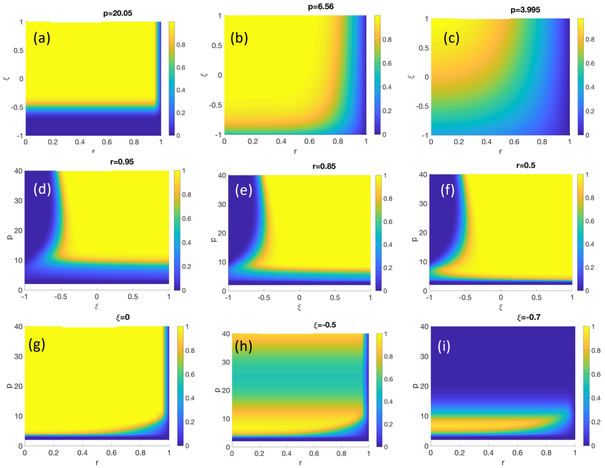
- An advantage of the probabilistic approach is that P_{RE} is a kind of **Green's function** for the RE seed density computation
- That is, once P_{RE} is computed, the RE seed production can be directly evaluated for **any** initial condition, f_0, by simply doing the integral

$$n_{RE}(t) = \int P_{RE}(\mathbf{r}, \mathbf{p}, t)\, f_0(\mathbf{r}, \mathbf{p})\, d\Omega$$

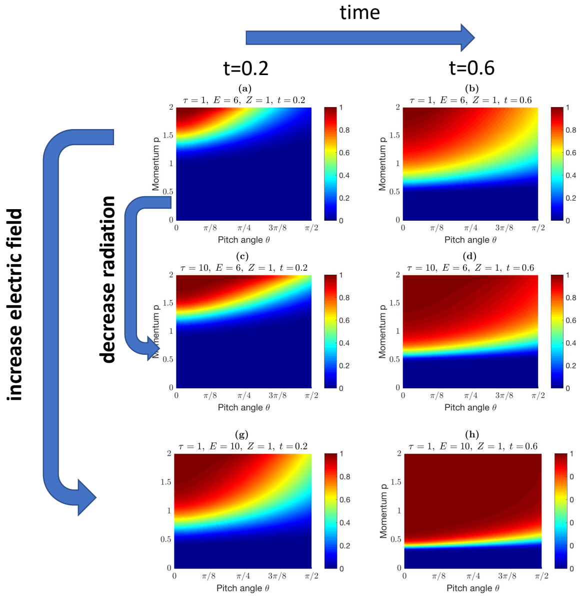
- This allows the **fast evaluation of different i.c. scenarios**
- On the other hand, in the standard approach, for each initial condition, $f_0(\mathbf{r}, \mathbf{p})$, we have to solve the whole time-dependent Fokker-Planck initial value problem to get $f(\mathbf{r}, \mathbf{p}, t)$

$$n_{RE}(t) = \int_{\Omega_{RE}} f(\mathbf{r}, \mathbf{p}, t)\, d\Omega$$

- In the probabilistic approach, the runaway region $\Omega_{RE}(\mathbf{r}, \mathbf{p}, t)$ corresponds to the region where $P_{RE}(\mathbf{r}, \mathbf{p}, t) \sim 1$
- In 3D (p, ζ, r), the boundary of $\Omega_{RE}(\mathbf{r}, \mathbf{p}, t)$ is not sharp and its time-dependent shape can be highly nontrivial.
- This can be problematic for analytical studies based on the standard approach that assume a simple shape of Ω_{RE}



THE PROBABILISTIC APPROACH ALLOWS THE EFFICIENT EXPLORATION OF DIFFERENT DISRUPTION SCENARIOS



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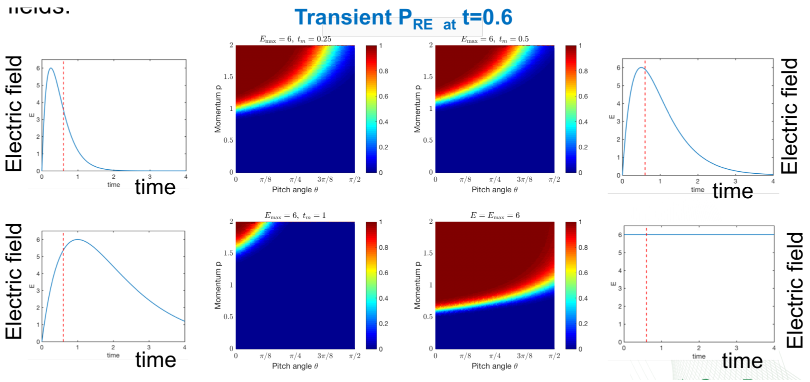
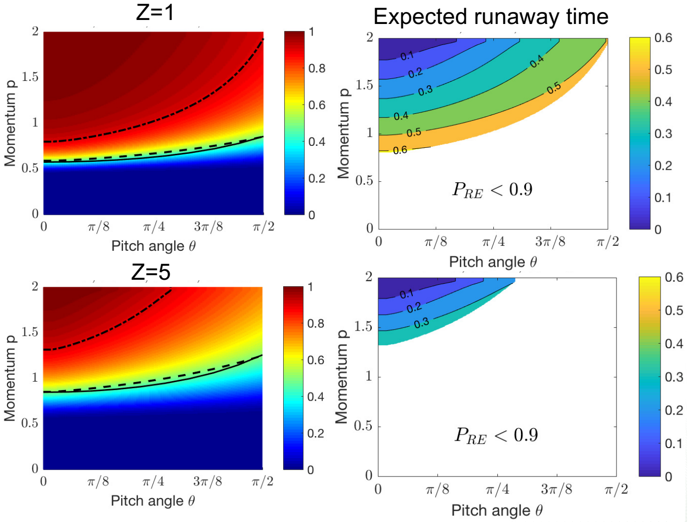


THE PROBABILISTIC APPROACH ALSO PROVIDES THE EXPECTED RUNAWAY TIME



HOW TO COMPUTE THE PROBABILITY OF RUNAWAY P_{RE}?

- **Direct Monte-Carlo** [Fernandez-Gomez, et al.,Phys. Plasmas (2012)]. Straightforward to implement but inefficient and potentially inaccurate due to statistical sampling errors.
- **Adjoint Fokker-Planck** [Liu, et al., Phys. Plasmas (2016)]. Elegant and more efficient than the direct MC, but it requires the numerical solution of a PDE.
- **Backward-Monte Carlo** [Zhang et al., Phys. Plasmas (2017); Yang et al., Journal Comp. Phys. (2021)]. Based on the Feynman-Kac formula. Reduces the problem to the computation of Gaussian integrals. No MC sampling or PDE solving required! Efficient and unconditionally stable.

PROPOSED FEYNMAN-KAC BASED METHOD
(Simple version)

- To simplify the discussion, consider the following pitch angle, ξ, and momentum, p, Fokker-Plank model

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial p}(b_1 f) + \frac{\partial}{\partial \xi}(b_2 f) - \frac{1}{2}\frac{\partial^2}{\partial \xi^2}(\sigma^2 f) = 0$$

- In this case $P_{RE}(T-t,p,\xi) = P(t,p,\xi)$ where $P(t,p,\xi)$ is the solution of adjoint FP which according to the Feynman-Kac formula is given by the conditional expectation

$$P(t,p,\xi) = \mathbb{E}[\chi(p_T,\xi_T)\,|\,p_t = p, \xi_t = \xi]$$

$$\chi(p_T,\xi_T) = \begin{cases} 1, & \text{if } \ p_T \geq p_*, \\ 0, & \text{otherwise}, \end{cases}$$

where p_t and ξ_t are the paths of the stochastic equations

$$\begin{aligned} dp_t &= b_1(p_t,\xi_t)\,dt, \\ d\xi_t &= b_2(p_t,\xi_t)\,dt + \sigma(p_t,\xi_t)\,dW_t \end{aligned}$$

DISCRETIZATION OF FEYNMAN-KAC FORMULA REDUCES THE COMPUTATION TO GAUSSIAN INTEGRALS

- Introduce a partion $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_N = T\}$, of $[0, T]$, and for small $\Delta t = t_{n+1} - t_n$ approximate

$$\begin{aligned} p_{t_{n+1}} &\approx p_{t_n} + b_1(p_{t_n}, \xi)\,\Delta t \\ \xi_{t_{n+1}} &\approx \xi_{t_n} + b_2(p_{t_n}, \xi_{t_n})\,\Delta t + \sigma(p_{t_n}, \xi_{t_n})\,\Delta W, \end{aligned}$$

- Within the time interval $[t_n, t_{n+1}]$, write

$$P(t_n, p, \xi) = \mathbb{E}\left[P(t_{n+1}, p_{t_{n+1}}, \xi_{t_{n+1}}) \mid p_{t_n} = p, \xi_{t_n} = \xi\right].$$

  and, using the Gaussian propagator, approximate

$$P(t_n, p, \xi) \approx \int_{\mathbb{R}} P\left(t_{n+1}, p + b_1\Delta t, \xi + b_2\Delta t + \sigma x\right)\, \frac{e^{-\frac{1}{2}\frac{x^2}{\Delta t}}}{\sqrt{2\pi\Delta t}} dx\,,$$

  which can becomputed using Gauss-Hermite quadrature rules.
- Also, an interpolation in (ξ, p) space is needed at each step.

- Some advantages of the method
  - Unconditionally stable (no need to solve PDEs)
  - Second order convergence in space, first order in time
  - No need to sample orbit (no MC noise)
  - Straightforward to parallelize.

- Further details of the method, including:
  - GPU accelerated matrix representation implementation for time-dependent models, e.g. $T = T(t)$ and $E = E(t)$.
  - Use of piecewise cubic Hermite interpolating polynomials
  - 3D examples including applications to fluid mechanics
  - Benchmarks with analytical solutions and comparisons with explicit and implicit adjoint Fokker-Planck solvers

  can be found in:

  M.Yang, G. Zhang, D. del-Castillo-Negrete, and M.Stoyanov, "A Feynman-Kac based numerical method for the exit time probability of a class of transport problems." Accepted for publication in Journal of Computational Physics (2021).

3D RUNAWAY ELECTRON ACCELERATION MODEL

3D+1 Fokker-Planck equation for $f(r, p, \xi; t)$

$$\frac{\partial f}{\partial t} = \mathcal{F} + \mathcal{R} + \mathcal{C} + \mathcal{D}\,, \qquad \text{with}$$

- Electric field force $\mathcal{F}\{f\} = -E\left[\xi \frac{\partial f}{\partial p} + \frac{(1-\xi^2)}{p}\frac{\partial f}{\partial \xi}\right]$
- Synchrotron radiation reaction force

$$\mathcal{R}\{f\} = \frac{1}{\tau}\left\{\frac{1}{p^2}\frac{\partial}{\partial p}\left[p^3\gamma\left(1-\xi^2\right)f\right] - \frac{\partial}{\partial \xi}\left[\frac{1}{\gamma}\xi\left(1-\xi^2\right)f\right]\right\}.$$

- Collision operator

$$\mathcal{C}\{f\} = \frac{1}{p^2}\frac{\partial}{\partial p}\left\{p^2\left[C_A\frac{\partial f}{\partial p} + C_F f\right]\right\} + \frac{C_B}{p^2}\frac{\partial}{\partial \xi}\left[\left(1-\xi^2\right)\frac{\partial f}{\partial \xi}\right],$$

- Radial diffusion operator

$$\mathcal{D}\{f\} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial f}{\partial r}\right],$$

COLLISIONS MODEL

$$
\begin{aligned}
C_A(p) &= \bar{\nu}_{ee}\, \bar{v}_T^2\, \frac{\psi(x)}{x} \\
C_B(p) &= \frac{1}{2}\, \bar{\nu}_{ee}\, \bar{v}_T^2\, \frac{1}{x}\left[Z + \phi(x) - \psi(x) + \frac{\delta^4}{2}x^2\right] \\
C_F(p) &= 2\, \bar{\nu}_{ee}\, \bar{v}_T\, \psi(x)\,.
\end{aligned}
$$

where $x = \frac{1}{\bar{v}_T}\frac{p}{\gamma}$, $\qquad \gamma = \sqrt{1 + \left(\tilde{\delta} p\right)^2}$, $\qquad \tilde{\delta} = \frac{\tilde{v}_T}{c} = \sqrt{\frac{2\tilde{T}}{mc^2}}$

$$
\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds\,, \qquad \psi(x) = \frac{1}{2x^2}\left[\phi(x) - x\frac{d\phi}{dx}\right]
$$

and the time dependence enters through the variables

$$
\bar{v}_T(t) = \sqrt{\frac{\hat{T}}{\tilde{T}}}\,, \qquad \bar{\nu}_{ee}(t) = \left(\frac{\tilde{T}}{\hat{T}}\right)^{3/2} \frac{\ln\hat{\Lambda}}{\ln\tilde{\Lambda}}\,, \qquad \delta(t) = \sqrt{\frac{2\hat{T}}{mc^2}}
$$

where $\hat{T}(t)$ denotes the time-dependent plasma temperature.

DIFFUSION, COOLING, AND ELECTRIC FIELD MODELS

- Rechester-Rosenbluth type radial diffusion model

$$D = \hat{D}_0\, F(r) G(p)\,, \qquad \hat{D}_0 = \pi q v_{\|} R \left(\frac{\delta B}{B}\right)^2 ,$$

with spatial and momentum dependence

$$F(r) = \frac{1}{2}\left\{1 + \tanh\left[\frac{r - r_D}{L_D}\right]\right\}, \qquad G(p) = e^{-(p/\Delta p)^2} .$$

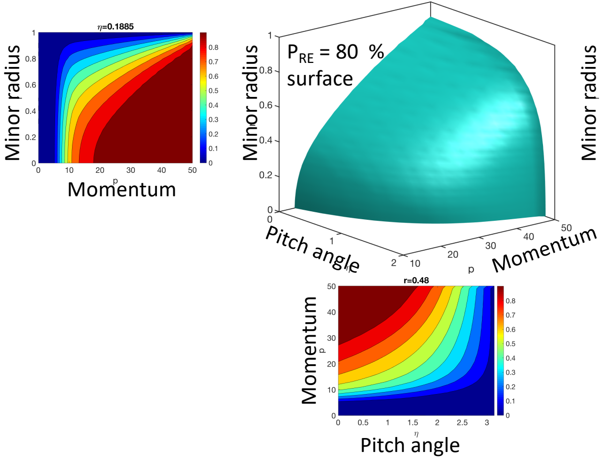
- Exponential cooling model with thermal quench time scale t_*

$$\hat{T} = \hat{T}_f + \left(\hat{T}_0 - \hat{T}_f\right) e^{-t/t_*} ,$$

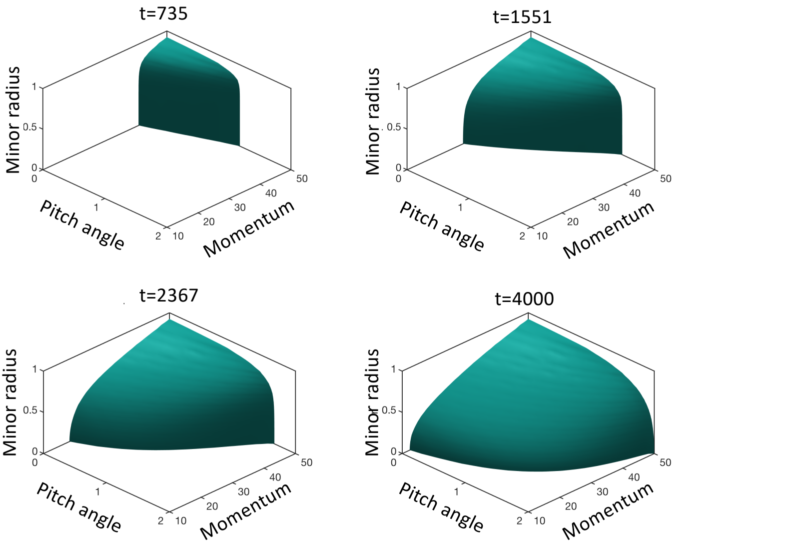
- Electric field dependence from Ohms's law and Spitzer conductivity

$$E(t) = E_0 \left[\frac{\hat{T}_0}{\hat{T}(t)}\right]^{3/2} .$$

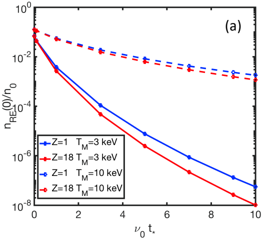
PROBABILITY OF RUNAWAY IN 3-DIMENSIONS
Momentum × Pitch Angle × Minor Radius space



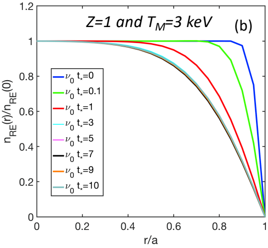
TIME EVOLUTION OF 3-D 80% PROBABILITY OF RUNAWAY ISO-SURFACE

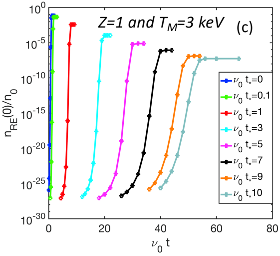


DEPENDENCE OF SEED RUNAWAY ELECTRONS ON THERMAL QUENCH TIME SCALE



- Seed production has a strong dependence on thermal quench time and initial temperatura
- There is a weaker dependence on Z
- Diffusion reduces the gradient of the radial seed density profile at the edge and this effect increases with the termal quench time
- If the electric field does not grow fast enough, radial diffusion will deconfine the electrons before they can be accelerated
- The onset of the saturation of the seed runaway population is significantly affected by the duration of the thermal quench.



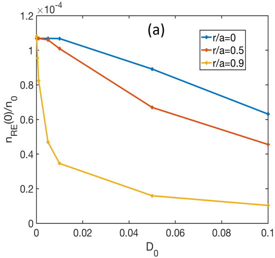


Time normalized by $1/n_0$ =0.1 ms

$D_0=10^2\ m^2/s$ ($\sim\delta B/B \sim 10^{-4}$)
$\Delta p=2$
$E_0=2 \times 10^{-3}$ V/m
$T_0=3$ keV $T_f=0.005$ keV
$\tau=6000$
$P_{min}=1$ $P_{max}=15$ (1 MeV)

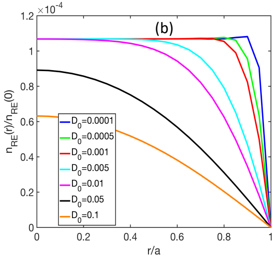
RE_Production_t_star_r_dependence_03122021.m

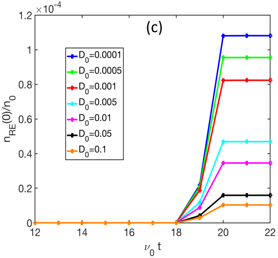
DEPENDENCE OF SEED RUNAWAY ELECTRONS ON RADIAL DIFFUSIVITY



- As expected, seed production decreases when D_0 *increases and this effect is more noticiable near the edge*
- Increasing the radial diffusivity leads to the "flattening" of the seed production rate profile
- However, the onset of the saturation of the seed runaway population is not affected by the value of D_0.

D_0 normalized by 10^4 m^2/s





$1/\nu_0$ =0.1 ms

$\nu_0\, t_*=3$

$T_M=3$ keV

$Z=1$

$\Delta p=2$

$E_0=2 \times 10^{-3}$ V/m

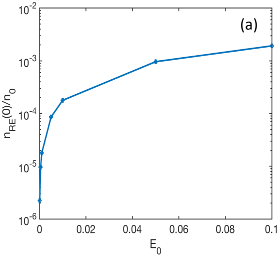
$T_0=3$ keV $T_f=0.005$ keV

$\tau=6000$

$P_{min}=1$ $P_{max}=15$ (1 MeV)

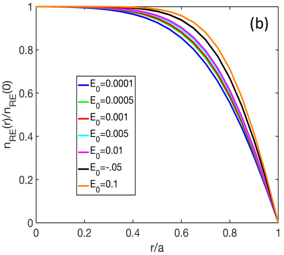
RE_Production_Dr_r_dependence_04042021.m

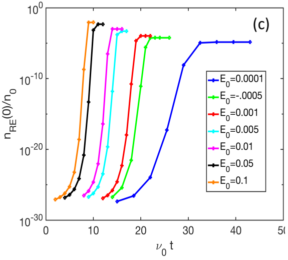
DEPENDENCE OF SEED RUNAWAY ELECTRONS ON ELECTRIC FIELD



- Seed production exhibits the typical increase with E_0
- As the electric field decreases the profiles become slightly shallow due to the loss of confinement before runaway acceleration
- The electric field can also significantly delay the onset of the seed RE generation

E_0 normalized by $E_D/2$~ 2 V/m





$1/\nu_0$ =0.1 ms
$\nu_0\, t_*=3$
$T_M=3$ keV
$Z=1$
$D_0=10^2\ m^2/s$ (~$\delta B/B$ ~10^{-4})
$\Delta p=2$
$T_0=3$ keV $T_f=0.005$ keV
$\tau=6000$
$P_{min}=1$ $P_{max}=15$ (1 MeV)

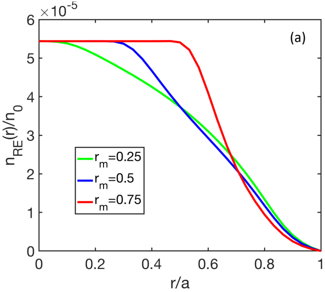
RE_Production_E_r_dependence_04052021.m

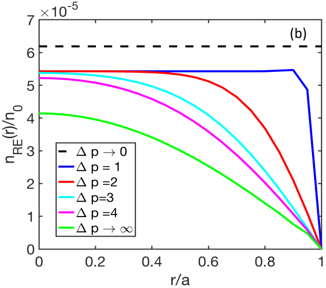
DEPENDENCE OF SEED RUNAWAY ELECTRONS ON MOMENTUM AND RADIAL DEPENDENCE OF DIFFUSIVITY

As a simple model of the increased stochasticity at the edge and the suppression of diffusion for high energy RE we consider

$$D(r,p) = \frac{D_0}{2}\left\{1 + \tanh\left[\frac{r - r_D}{L_D}\right]\right\} e^{-(p/\Delta p)^2}$$

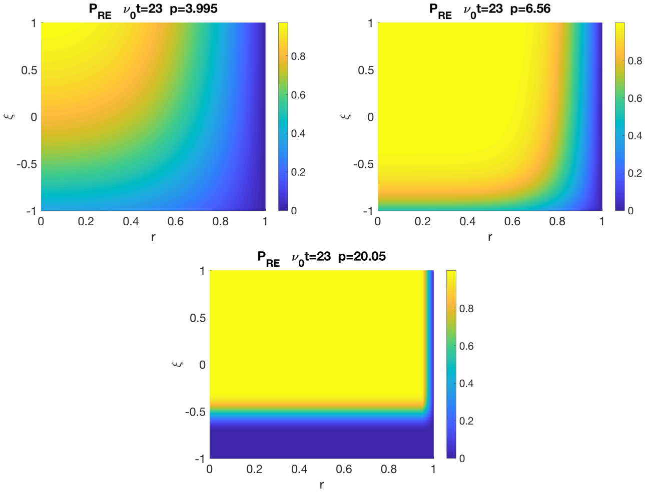
where D_0 is the Rechester-Rosenbluth diffusivity





HIGH MOMENTUM SUPRESION OF RADIAL DIFFUSIVITY

Probability of runaway as function of (r, ζ) at fixed p.



CONCLUDING REMARKS

- The probabilistic approach shifts attention from the computation of the RE distribution (e.g., solving the FP eq) to the computation of the probability of runaway P_{RE}
- The P_{RE} "maps" provide by themselves a unique insight on the RE generation process
- When it comes to the computation of the production rate, the P_{RE} acts as a Green's function from which the evolution of the seed density can be explored for many different initial conditions
- As a bonus, the probabilistic approach also provides the expected runaway time
- However the computation of the P_{RE} is not trivial and could be hard/time consuming
- Here we presented current progress on our approach based on the use of the Feynman-Kac formula

CONCLUDING REMARKS

- The Feynman-Kac formula approach (also known as the Backward Monte Carlo method) reduces the problem to the computation of Gaussian integrals that can be computed efficiently using Gauss-Hermite quadrature algorithms
- There is also the need for interpolation that we do using picewise cubic Hermite interpolation polynomials.
- A GPU accelerated matrix representation was implemented to compute the entire time evolution of the exit time probability using a single pass of the algorithm.
- We discussed applications to different disruption scenarios in 3D+time (radius, pitch angle and momentum)
- Of particular interest was the dependence of the production rate on confinement losses cause by magnetic field stochasticity that we modeled we radial diffusion
- Future work includes more detailed orbit dynamics and dynamic coupling to the plasma state [Hirvijoki, et al., Phys. of Plasmas 25, 062507 (2018)].