# A Theoretical Model for the Penetration of a Shattered-Pellet Debris plume

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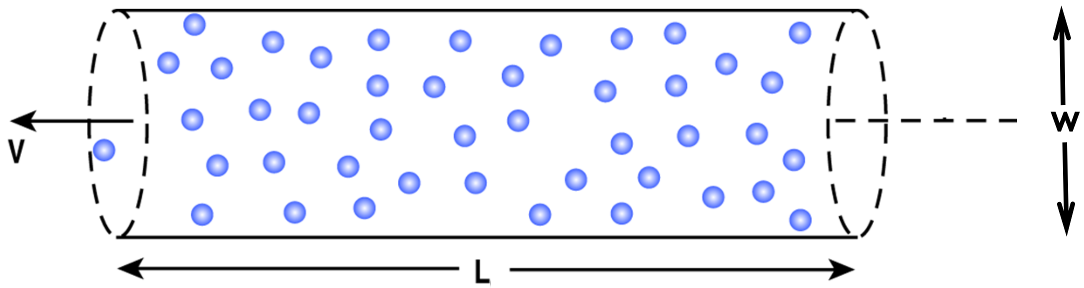
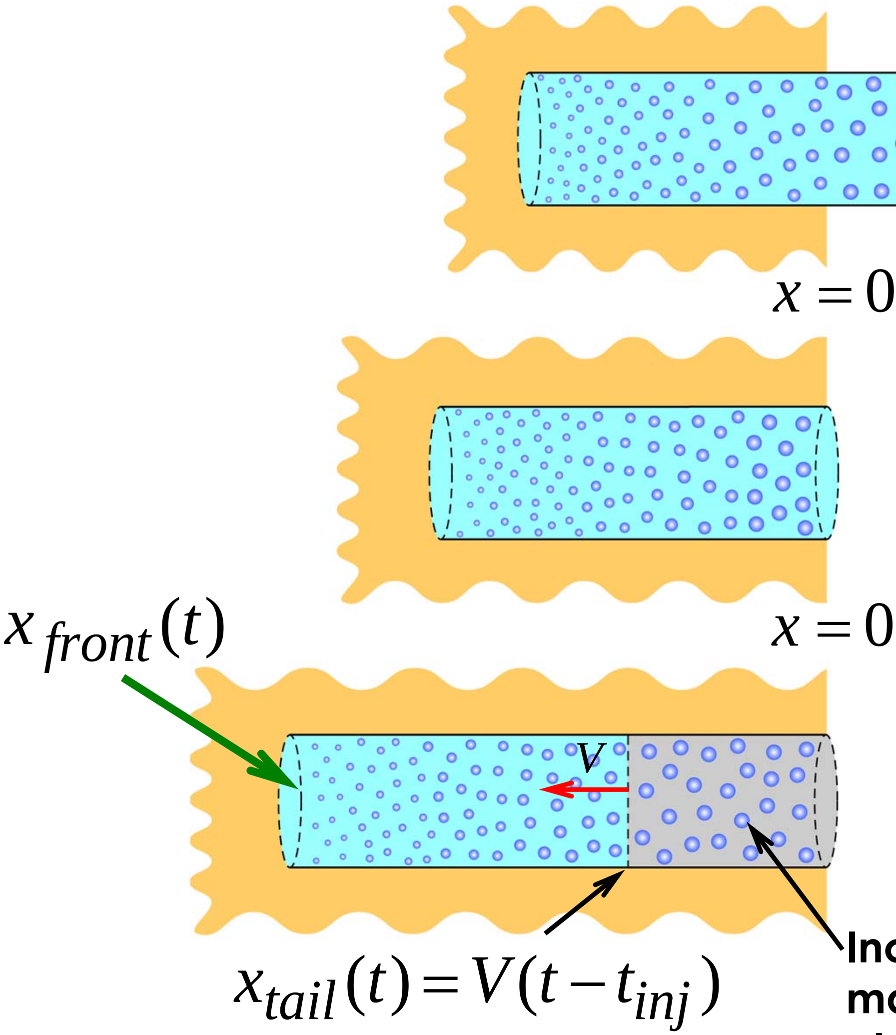
# Shattered pellet fragments form a Debris Plume

- Simple rigid beam model: blunt cylindrical shape, uniformly distributed pellets all with same size and velocity V.




- SPI drift tube diameter in ITER is $D_{tube} = 4 \text{ cm}$
- Due to divergence, mean plume diameter downstream is larger, say $w = 30 \text{ cm}$
- Total Injection time from 2016 Debris Plume Theory $t_{inj} = 0.6 \text{ ms}$ for V =500 m/s
- Plume length $L = V \cdot t_{inj} = 30 \text{ cm}$ for for V =500 m/s

# Stages of Propagation




**Attached plume $(0 < t < t_{inj})$**

**$t = t_{inj} = L/V$**

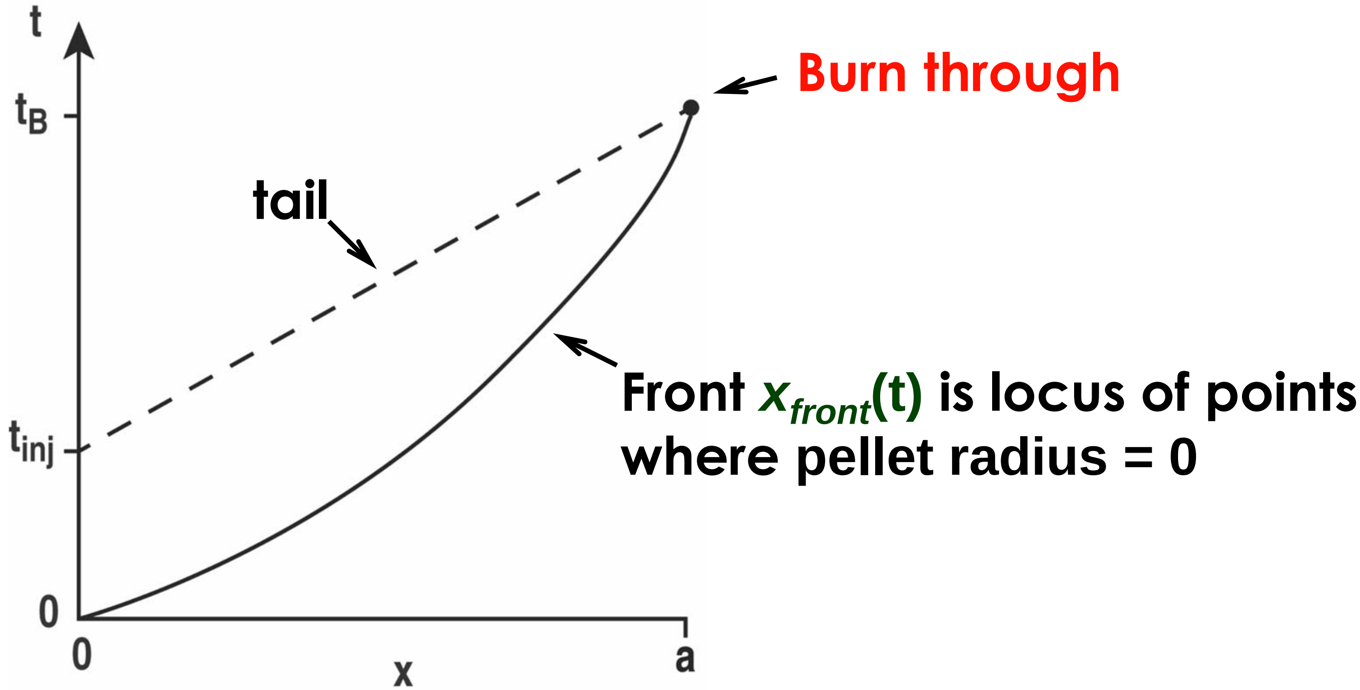
**injection time**

**Detached stream $(t > t_{inj})$**

**Include “virtual” section to ensure mathematically continuous BC at plasma edge**

# Find Trajectory of Moving Plume Front

**When boring through plasma, the plume front moves slower than the original plume speed *V*, "pencil sharpening effect".**




- **Ideal assimilation $x_{front} = a$, when the tail catches up with the front at the magnetic axis giving burnthrough time**

$$t_B = t_{inj} + a/V$$



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# Kinetic Model of Ambient Plasma Cooling

- **Pellets ablate and deposit cold ionized ablation trail which expands along magnetic field and radiates.**

- **The ionized ablation material is tenuous enough to allow inter penetration of hot ambient plasma electrons Proof!**

  - **Columnar density of the ionized impurities remains constant while expanding along the magnetic field**

$$\Sigma_{\parallel} = \int_{-\infty}^{\infty} n_I \, ds = \text{constant}$$

  - **If all pellet fragments ablate fully such that impurities are distributed evenly across minor radius then from mass conservation** $\Sigma_{\parallel} = N_I / wak$

$N_I = \text{number of neon atoms deposited, } k = \text{number of injectors}$

  - **Electrons streaming through plume suffer only a small collisional energy loss**

$$\frac{\Delta E}{E} = \frac{\Sigma_{\parallel}}{E} L(E) << 1 \quad L(E) = \frac{2\pi e^2 Z_a}{E} \ln\left[\frac{E}{I_*}\left(\frac{e}{2}\right)^{1/2}\right] \quad \text{(Bethe stopping power, } I_* = 135.5 \text{ eV for Ne)}$$

  - **This means pellet fragments are bathed in a two-temperature plasma: Hot ambient electrons and freshly ionized cold electrons. Only hot electrons do the ablating. How fast do they cool?**

# Ambient Plasma Cooling Contd

- **Kinetic equation describes evolution of plasma electron distribution function due to inelastic collisions with impurity atoms/ions)**

$$\frac{\partial f}{\partial t} = \frac{\partial}{\upsilon^2 \partial \upsilon}\left(\upsilon^2 a_{drag} f\right) \qquad a_{drag} = \frac{\langle n_I \rangle}{m_e} L(E) \qquad E = m_e \upsilon^2 / 2$$

$$\langle n_I \rangle = \text{flux - surface - averaged impurity nuclei density}$$

- **Pellet ablation rate depends on electron temperature of a Maxwellian plasma. Assume bulk electrons remain roughly Maxwellian:**

$$f(\upsilon) \approx f_{\max} = n_e \left(\frac{m_e}{2\pi T_e(t)}\right)^{3/2} \exp\left(-\frac{m\upsilon^2}{2T_e(t)}\right)$$

- **Take energy moment of kinetic equation to get**

$$\left.\frac{\partial T_e}{\partial t}\right)_x = -5.812 \times 10^{-6} \frac{\langle n_I \rangle}{T_e^{1/2}} Z_a \ln\left(\frac{T_e}{1.528 I_*}\right)$$

- **Use simpler approximation and generalize to neon-deuterium mixtures**

$$\left.\frac{\partial T_e}{\partial t}\right)_x = -2.2 \times 10^{-6} \frac{\langle n_{Ne} \rangle}{T_e^{1/4}}\left(Z_{Ne} + \frac{2X}{1-X}\right) \qquad X = \frac{\text{mol } D_2}{\text{mol } N_e + \text{mol } D_2}$$

# Independent Pellet Ablation Model

- **Each pellet ablates as though it were isolated from the rest**
  - **Obscuration of $||$-electron flux by its fellow fragments is typically small**

$$\Delta q_{||} / q_{||} = \tau_{\text{pell}} << 1$$

**number concentration of pellets**

  - **Optical depth** $\tau_{\text{pell}} = n_{\text{pell}} \pi r_p^2 w$

$$n_{\text{pell}} = \frac{\text{Total Mass added}}{\text{Mass per pellet}}$$

$$\tau_{\text{pell}} = \frac{3 \cdot \text{SDP}}{4 \rho_0(X) r_p}$$

**Analogous to $\tau_{\text{cloud}}$ for scattering of sunlight by cloud water droplets, replacing LWP → SDP**

$\rho_0(X)$ : **pellet density (g/cm$^3$)**

**Solid Debris Path (g/cm$^2$)**

- **High level of solid pellet transparency ( even more transparent than gas )**

$$N_{Ne} = 0.041 \text{ moles}, \; k = 2, \; L = 30 \text{ cm}, \; w = 30 \text{ cm}, \; r_p = 0.1 \text{ cm}, \; X = 0, \; \rho_0 = 1.444$$

$$\text{SDP} = 0.0004 \text{ g/cm}^2 \longrightarrow \tau_{\text{pell}} = 0.0024$$

# A Practical Expression for the Ablation Rate of Composite Neon-Deuterium Pellets §

$$G = \lambda(X)\left(\frac{T_e}{2000}\right)^{5/3}\left(\frac{r_p}{0.2}\right)^{4/3} n_{e14}^{1/3}$$

$G$(g/s) $T_e$(eV)
$r_p$(cm) $n_e$($10^{14}$cm$^{-3}$)

$$\lambda(X) = 27.0837 + \mathrm{Tan}[1.48709X]$$

- **Molar deposition rates per pellet**

$$\frac{dN_{Ne}}{dt} = \frac{(1-X)G}{W_{Ne}(1-X)+XW_{D_2}} \qquad \frac{dN_{D_2}}{dt} = \frac{XG}{W_{Ne}(1-X)+XW_{D_2}}$$

$$W_{Ne} = 20.183 \quad W_{D_2} = 4.0282 \quad \text{(g/mol)}$$

- **Pellet surface recession speed** $\dot{r}_p = -G/(4\pi r_p^2 \rho_0)$

$$\frac{Dy}{Dt} = -3.572\times10^{-6}\,\frac{\lambda(X)}{r_0^{5/3}\rho_0(X)}\,T_e^{5/3} n_{e14}^{1/3} \qquad y = (r_p / r_0)^{5/3}$$

§ Parks, to be published

# Flux-Surface-averaged gas density build up

- **Build up rate of neon atoms on a magnetic flux shell of differential thickness** $\delta x_\psi$

$$\delta \dot{N}_{Ne} = n_{pell} \cdot \frac{dN_{Ne}}{dt} \cdot w^2 \delta x_\psi$$

  - **Flux shell volume** $\delta V_\psi = 2\pi R \cdot 2\pi r \delta x_\psi$

- **Flux-averaged neon density increase** $\frac{\partial \langle n_{Ne} \rangle}{\partial t} = \frac{\delta \dot{N}_{Ne}}{\delta V_\psi}$

$$\longrightarrow \quad \frac{\partial \langle n_{Ne} \rangle}{\partial t} = \frac{N_{Ne} A}{4\pi^2 L R r}\left(\frac{3G}{4\pi r_0^3 \rho(X)}\right) \qquad \frac{\partial \langle n_D \rangle}{\partial t} = \frac{2X}{1-X} \frac{\partial \langle n_{Ne} \rangle}{\partial t}$$

$A$: Avogadro's number

# Coupled System of PDEs Describes 1-D SPI Dynamics

- **Independent variables** $(\xi, t)$ $\quad \xi = x/a = \text{streamwise distance}$

- **Pellet radii change**

$$\frac{\partial y}{\partial t} + \frac{V}{a}\frac{\partial y}{\partial \xi} = -\frac{\Theta^{5/3} n_{e14}^{1/3}(\xi)}{t_{abl}}$$

- **Plasma Cooling**

$$\frac{\partial \Theta^{5/4}}{\partial t} = -\frac{\tilde{n}}{t_{cool}}$$

- **Flux-averaged neon density rise**

$$\frac{\partial \tilde{n}}{\partial t} = \frac{9}{5}\frac{y^{4/5}\Theta^{5/3} n_{e14}^{1/3}(\xi)}{(1-\xi)t_{abl}}$$

$$y(\xi,t) = \left(r_p(\xi,t)/r_0\right)^{5/3}$$

$$\Theta(\xi,t) = T(\xi,t)/T_0$$

$$\tilde{n}(\xi,t) = \langle n_{Ne} \rangle / n_{\max}$$

$$n_{\max} = \frac{N_{Ne}A}{4\pi^2 LRa}, \quad T_0 = T(a,0)$$

- **Characteristic time constants:**

$$t_{abl} = 2.8\times 10^5 \left(\frac{r_0}{T_0}\right)^{5/3} \frac{\rho_0(X)}{\lambda(X)}$$

(Ablation time)

$$t_{cool} = 3.63\times 10^5 \frac{T_0^{5/4}}{n_{\max}}\left(Z_{Ne} + \frac{2X}{1-X}\right)^{-1}$$

(Cooling time)

# Insight From a 0-D Semi-Analytical Solution

- **Assume Plume L = a is deposited in plasma instantly at t = 0** $\partial / \partial \xi = 0$

$$\tilde{n} = 1 - y^{9/5} \qquad \Theta = \left[1 - d\left(1 - \frac{14}{9}y + \frac{5}{9}y^{14/5}\right)\right]^{12/35}$$

$$d = \frac{3t_{abl}}{2t_{cool}} \propto \frac{N_{Ne} r_0^{5/3}}{T_0^{35/12}}$$

$$\frac{dy}{dt} = -\frac{\omega(y)}{t_{abl}}, \qquad \omega(y) = \left[1 - d\left(1 - \frac{14}{9}y + \frac{5}{9}y^{14/5}\right)\right]^{4/7}$$

- **"Super-critical" injection d > 1: Plasma Cooling is so fast that temperature quench happens before pellets fully ablate**

$$\Theta \to 0\ , \quad y \to y_{crit}, \quad \tilde{n} \to 1 - y_{crit}^{9/5}, \quad t_{quench} \to t_{abl} \int_{y_{crit}}^{1} \omega(y)^{-1}\, dy$$

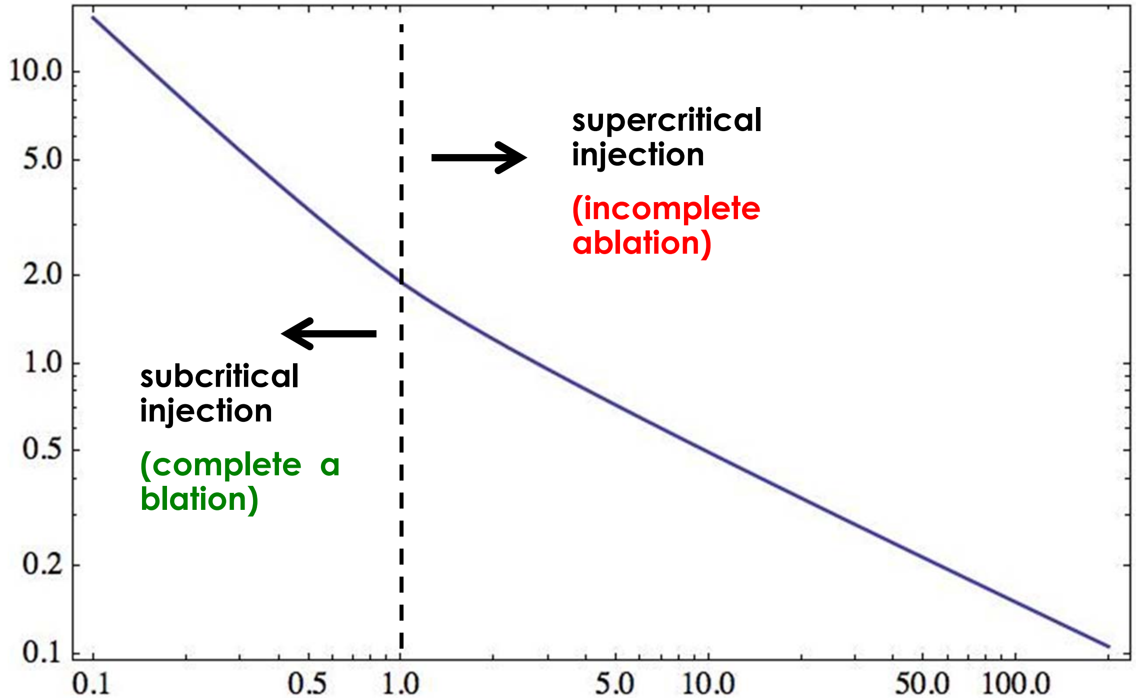
$$1 - (14/9) y_{crit} + (5/9) y_{crit}^{14/5} = d^{-1}$$

- **"Sub-critical" injection d < 1: Pellets "burn out" before temperature quench**

$$\text{STAGE 1:} \quad \Theta \to (1-d)^{12/35}\ , \quad y \to 0, \quad \tilde{n} \to 1 \qquad t \to t_* = t_{abl} \int_0^1 \omega(y)^{-1}\, dy$$

$$\text{STAGE 2:} \quad \Theta \to 0\ , \qquad t_{quench} \to t_* + t_{cool}(1-d)^{3/7}$$

# Quench Time Versus d




$$\frac{t_{quench}}{t_{abl}}$$

$$d$$

$$d = \frac{3t_{abl}}{2t_{cool}} \propto \frac{N_{Ne} r_0^{5/3}}{T_0^{35/12}}$$

- **Examples:** $X = 0$ (pure neon), $r_{p0} = 0.15$ cm, $N_{Ne} = 0.041$ moles

$t_{abl} = 0.136$ ms and $d = 1.676$ and $t_{quench} = 0.183$ ms for $T_0 = 10$ keV

$t_{abl} = 0.0693$ ms and $d = 0.514$ and $t_{quench} = 0.228$ ms for $T_0 = 15$ keV

# Transformation to 1-D Lagrangian Variables

$(x,t) \to (x,q) \qquad q = t - \dfrac{x}{V}$

**Lagrange coordinate q labels elemental slice of debris plume in motion**

- **The "first arrivals" enter the plasma at t = 0 and x = 0 with Lagrangian label** $q = 0$

- **The tail pellets enter last with Lagrangian label** $q = t_{inj}$

- **Normalized coordinates**

$$(x,q) \to (\xi,\zeta) \qquad \xi = x/a \quad (0 < \xi < 1) \qquad \zeta = q/t_{inj} \quad (0 < \zeta < 1)$$

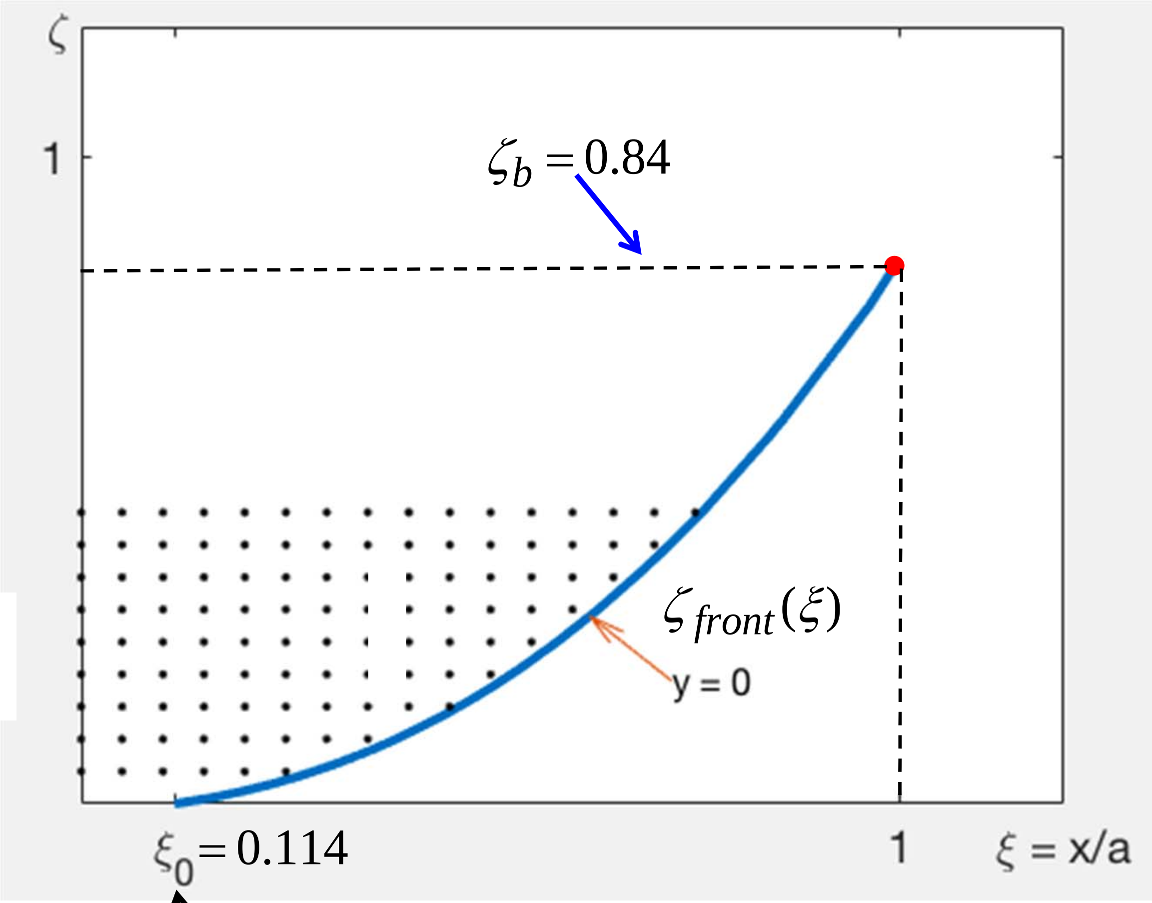
- **Additional time scales**

$$t_{inj} = \frac{L}{V}, \quad t_{transit} = \frac{a}{V}$$

# Transformed Equations Leads to a Cauchy-like Problem

- **Pellet radii change** $\left.\frac{\partial y}{\partial \xi}\right)_{\zeta} = -\sigma_1 \tilde{\Theta}^{4/3}$ $\sigma_1 = \frac{t_{transit}}{t_{abl}}$
- **Plasma Cooling** $\left.\frac{\partial \tilde{\Theta}}{\partial \zeta}\right)_{\xi} = -\sigma_2 \tilde{n}$ $\sigma_2 = \frac{t_{inj}}{t_{cool}}, \quad \tilde{\Theta} = \Theta^{5/4}$
- **Flux-averaged neon density rise** $\left.\frac{\partial \tilde{n}}{\partial \zeta}\right)_{\xi} = \sigma_3 \frac{y^{4/5}\tilde{\Theta}^{4/3}}{(1-\xi)}$ $\sigma_3 = \frac{9t_{inj}}{5t_{abl}}$
- **In these equations we assumed a flat density profile with** $n_{e14}(\xi) = 1$
- **Cauchy data:** **along the** $\zeta$**–axis** $y = 1$
  **along the** $\xi$**–axis** $\tilde{n} = 0$ and $\tilde{\Theta} = \left(T_e(\xi)/T_0\right)^{5/4}$ (← **initial $T_e$ profile**)

# Numerical Solution of Front Trajectory in Hodograph Plane




First arrivals burn out here

**The front trajectory intercepts the magnetic axis with**

$$\zeta = \zeta_b = 0.84$$

**This means 84% of solid debris plume was annihilated, with 16% left only partially ablated**

**The time for $\zeta_b$ element to reach the magnetic axis is**

$$t_b = \frac{a}{V} + \zeta_b t_{inj} = 4.24 \text{ ms}$$

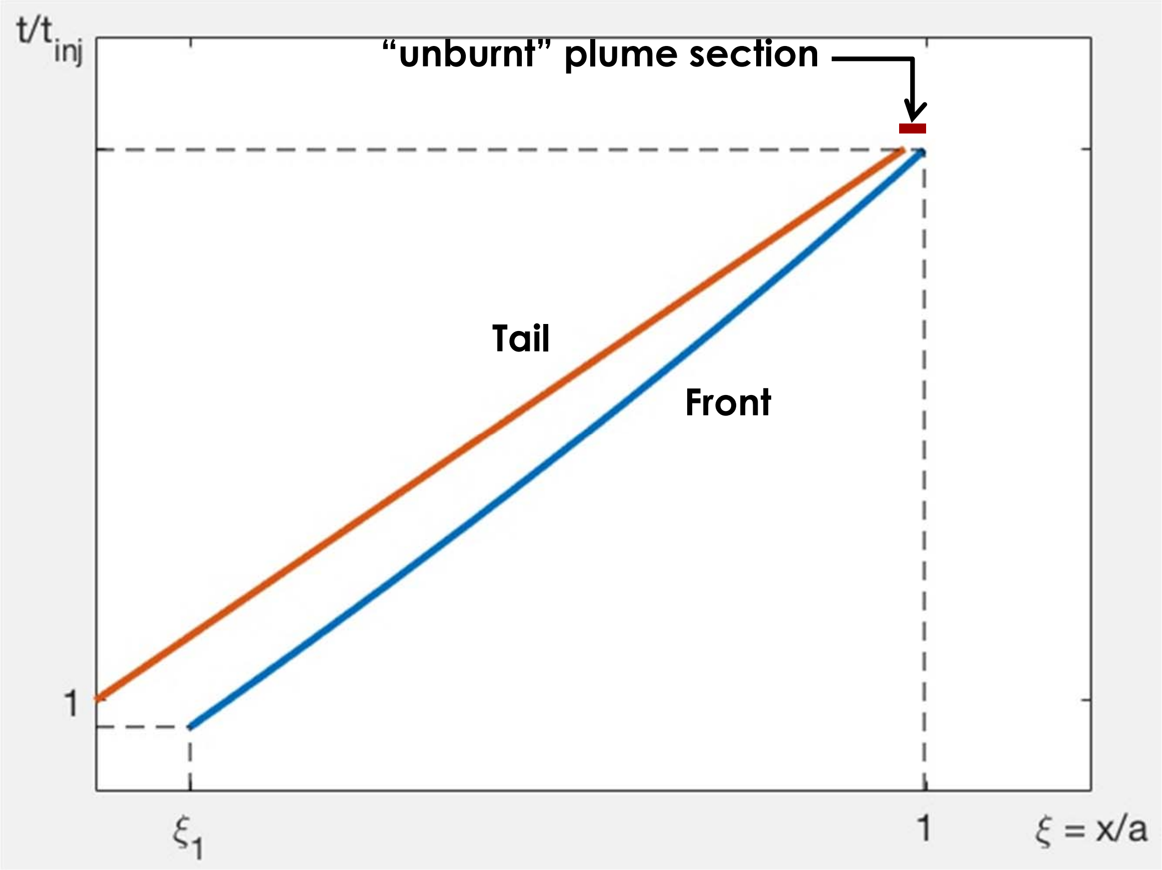
**For $t > t_b$ the surviving plume elements cross magnetic axis with no further ablation**

- **Parameters** $N_{Ne} = 0.041$ moles, $X = 0.5$, $r_{p0} = 0.15$ cm, $T_0 = 19.5$ keV,

  $L = 30$ cm, $a = 187$ cm, $V = 500$ m/s, $t_{inj} = 0.6$ ms.

# Numerical Solution of Front Trajectory in (x,t) Plane

- **Front has smaller velocity than tail velocity V due to erosion (pencil sharpening effect)**




- **To assimilate entire plume when front hits magnetic axis we could reduce d**

$$d \propto \frac{N_{Ne} r_0^{5/3}}{T_0^{35/12}}$$

- **e.g. make smaller pellets, or**
- **Reduce V or L**
- **Numerical iteration**

# Dilution Cooling Model

- **Assume most electrons added are free (valid for lots of deuterium X~1**
  - **Pellet radii change** $$\frac{\partial y}{\partial t}+V\frac{\partial y}{\partial x}=-\frac{\Theta^{5/3}\tilde{n}^{1/3}}{t_{abl}}$$
  - **Flux-averaged electron density rise** $$\frac{\partial \tilde{n}}{\partial t}=\frac{9}{10}\frac{a}{L}\frac{\Delta n_e}{n_{e0}}\frac{y^{4/5}\Theta^{5/3}\tilde{n}^{1/3}}{\rho t_{abl}}$$

$$\tilde{n}=\frac{n_e(x,t)}{n_{e0}},\quad \rho=1-x/a \qquad t_{abl}=\frac{2.8\times 10^5}{n_{e014}^{1/3}}\left(\frac{r_0}{T_0}\right)^{5/3}\frac{\rho_0(X\approx 1)}{\lambda(X\approx 1)}$$

$$\Delta n_e=\frac{\text{added free electrons}}{\text{plasma volume}}$$

- **Dilution Cooling** $\Theta(x,t)\tilde{n}(x,t)=P(x)$ $\qquad P(x)=\dfrac{p_e(x)}{p_{e0}}$ normalized pressure
- **Eliminate** $\Theta(x,t)$ **from equations**

# Dilution Cooling Model cont'd

$$\tau = t/t_*, \quad s = x/Vt_*, \quad Z = \left(\frac{\tilde{n}}{\tilde{n}_*}\right)^{7/3} \qquad t_* = t_{abl}\tilde{n}_*^{4/3} \qquad \tilde{n}_* = \frac{21}{10}\frac{a}{L}\frac{\Delta n_e}{n_{e0}}$$

- **Reduced Equations**

$$\frac{\partial y}{\partial \tau} + \frac{\partial y}{\partial s} = -Z^{-4/7} P(s)^{5/3}$$

$$\frac{\partial Z}{\partial \tau} = y^{4/5} P(s)^{5/3} / \rho(s)$$

- **Convert to Lagrangian variables** $(s,\tau) \rightarrow (s,\varsigma) \quad \varsigma = \tau - s$

$$\left.\frac{\partial y}{\partial s}\right)_{\varsigma} = -Z^{-4/7} P(s)^{5/3}$$

$$\left.\frac{\partial Z}{\partial \varsigma}\right)_{s} = y^{4/5} P(s)^{5/3} / \rho(s)$$



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# Further Simplifying Transformations

$$\mathrm{H} = \frac{Z\rho}{P^{5/3}} \qquad u(s) = \int_0^s P(s')^{5/7}\rho(s')^{4/7}\,ds' \quad (=0 \text{ at plasma boundary } s=0)$$

- **With above definitions we get**

$$\left.\frac{\partial y}{\partial u}\right)_\varsigma = -\mathrm{H}^{-4/7} \qquad \left.\frac{\partial \mathrm{H}}{\partial \varsigma}\right)_u = y^{4/5}$$

- **Similarity variable** $\eta = \dfrac{\varsigma}{u^{7/4}}$ **converts PDEs to ODEs**

with $\Phi = \dfrac{\mathrm{H}}{u^{7/4}}$

$$\frac{dy}{d\eta} = \frac{4}{7\eta}\Phi^{-4/7} \quad \text{pellet radii}$$

→

$$\frac{d\Phi}{d\eta} = y^{4/5} \quad \text{density rise}$$

# Universal Solution for y and $\Phi$

- **Dependent variables are reduced pellet radii and electron density**

$$y = \left(\frac{r_p}{r_0}\right)^{5/3} \qquad \Phi = \frac{\mathrm{H}}{u(s)^{7/4}} = \frac{\rho}{P(\rho)^{5/3} F(\rho)^{7/4}} \left(\frac{Vt_*}{a}\right)^{7/4} \left(\frac{\tilde{n}}{\tilde{n}_*}\right)^{7/3}$$

**Where** $u(s)\left(\frac{Vt_*}{a}\right) = F(\rho) = \int_{\rho}^{1} P(\rho')^{5/7} \rho'^{4/7} \, d\rho' \quad (= 0 \text{ at plasma boundary } \rho = 1)$

- **Boundary Conditions:**

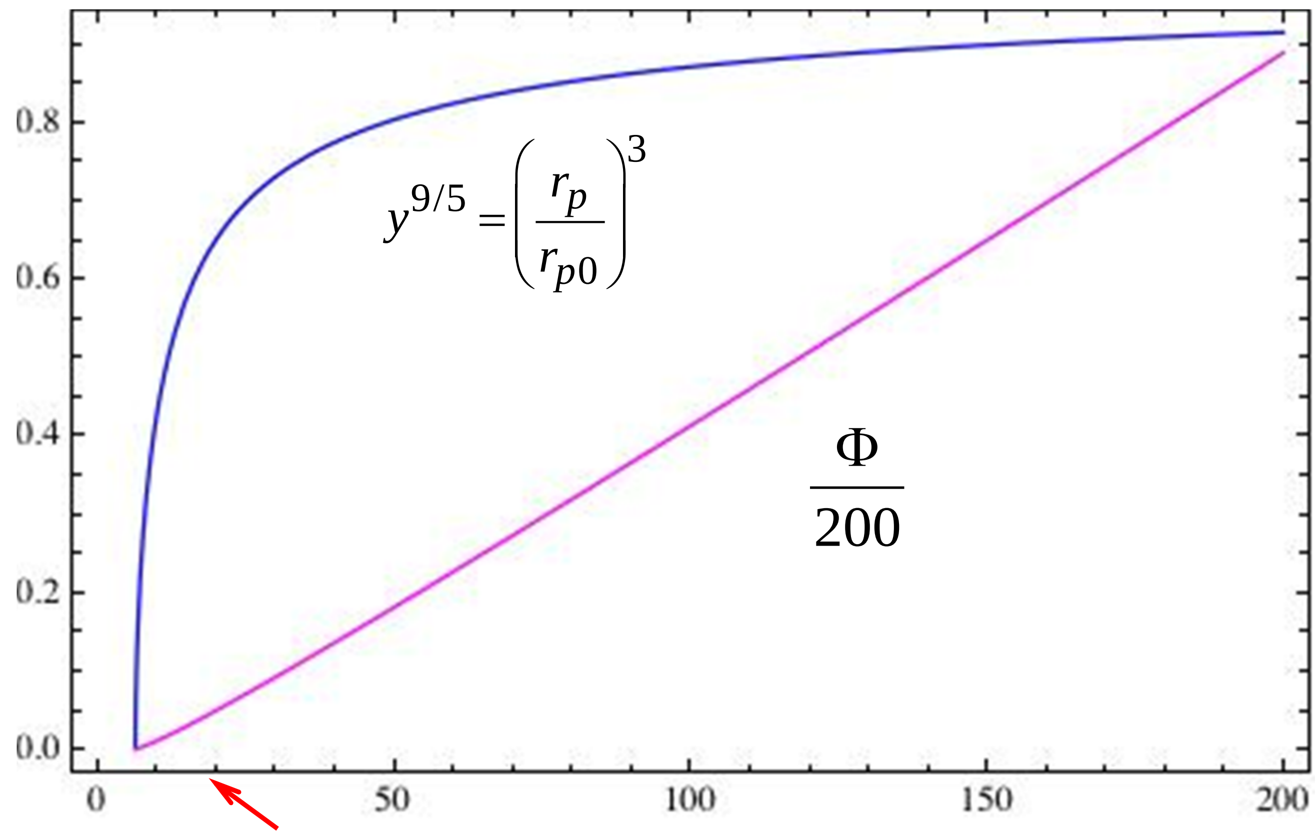
  ✓ $y = 1$ at plasma boundary $\eta = \varsigma / u^{7/4} \to \infty$

  ✓ $y = 0$ pellet radii = 0 at moving front $\eta = \eta_0$

  ✓ $\Phi = 0$ added density = 0 at moving front $\eta = \eta_0$

- **Isn't that 3 boundary conditions? No. Only 2 because the front position $\eta_0$ is not known *a priori*. We can only know $\eta_0$ by using a shooting method**

# Plot of Universal Solutions




$\eta$

$\eta_0 = 6.439$

**(numerical value of plume front position)**

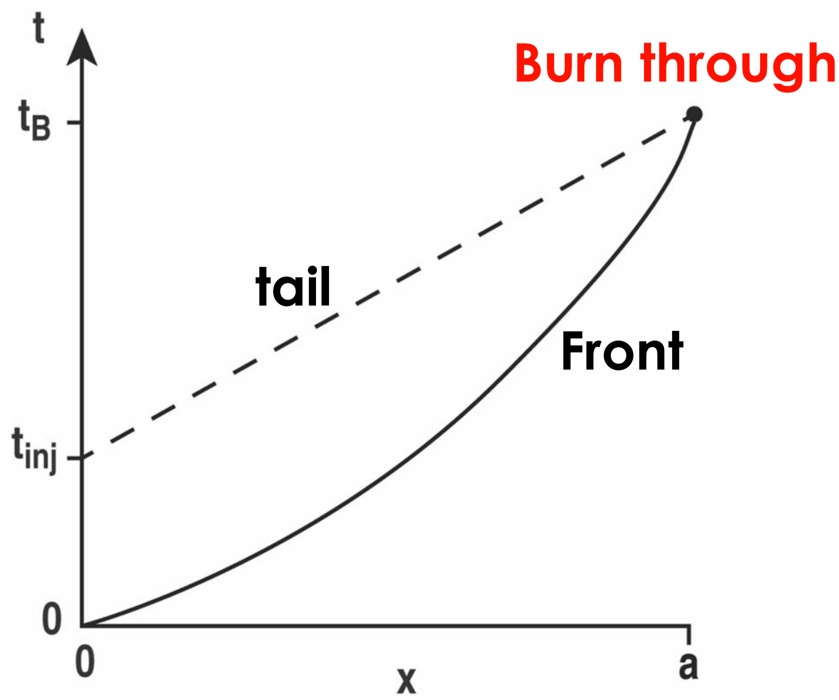
# Solution for Trajectory of Moving Debris Front

- **Front trajectory**

$$x_{front} = Vt - \eta_0 \frac{a^{7/4}}{(Vt_*)^{3/4}} F(1 - x_{front}/a)^{7/4}$$

- **Tail trajectory**

$$x_{tail} = V(t - t_{inj})$$




- **Optimized injection: Burn through when front and tail meet at the magnetic axis**

$$x_{tail} = x_{front} = a$$

$$t_B = t_{inj} + a/V$$

- **This leads to the optimum velocity…**

# Optimum Velocity

- **More added mass** → **more self-cooling** → **lower velocity**

$$V = \left(\frac{10\eta_0}{21}\right)^{4/3} F(0)^{7/3} \frac{a}{t_{abl}} \left(\frac{n_{e0}}{\Delta n_e}\right)^{4/3}$$

$$t_{abl} = \frac{2.8\times10^5}{n_{e014}^{1/3}} \left(\frac{r_0}{T_0}\right)^{5/3} \frac{\rho_0(X)}{\lambda(X)}$$

- **In ITER with** $\Delta n_e / n_{e0} = 30$ $\quad T_{e0} = 19.5\,\text{keV},\ a = 1.87\,\text{m},\ r_0 = 2\,\text{mm},\ F(0) = 0.34$

→

$$V = 1037\ \text{m/s} \quad \text{for } X = 1 \ \ (\text{pure } D_2)$$
$$V = 576\ \text{m/s} \quad \text{for } X = 0.9 \ \ (\text{mostly } D_2)$$
$$V = 210\ \text{m/s} \quad \text{for } X = 0.5$$

- **$E_{critical}$ ~ 5V/m , $E_{eff}$ ~ 10V/m , runaway beam decay time ~ 200 ms**

# Final Density Profile for Optimized Injection

- **Space-time electron density profile evolution**

$$n_e(x,t) = \frac{21\Delta n}{10\eta_0 F(0)} \frac{P^{5/7}(\rho)}{\rho^{3/7}} \left(\frac{F(\rho)}{F(0)}\right)^{3/4} \Phi^{3/7}\left[\eta_0 \left(\frac{t - x/V}{t_{inj}}\right)\left(\frac{F(\rho)}{F(0)}\right)^{-7/4}\right]$$

For $t < t_{inj}$ $0 < x < x_{front}(t)$ and for $t > t_{inj}$ $x_{tail}(t) < x < x_{front}(t)$

- **Density profile is "frozen in time" for $0 < x \leq x_{tail}(t)$. So final density profile obtained after burnout is found by setting $x = x_{tail}(t)$ in above expression:**

$$n_{efinal}(\rho) = \frac{21\Delta n_e}{10\eta_0 F(0)} \frac{P^{5/7}(\rho)}{\rho^{3/7}} \left(\frac{F(\rho)}{F(0)}\right)^{3/4} \Phi^{3/7}\left[\eta_0 \left(\frac{F(\rho)}{F(0)}\right)^{-7/4}\right] \qquad \Delta n_e = \frac{N_e}{2\pi R \pi a^2 \kappa}$$

- **Check for consistency: Does** $N_e = 4\pi R\kappa a^2 \int_0^1 n_{efinal}\rho\, d\rho$ **?**

$$1 = \frac{12}{5}\int_{\xi_0}^{\infty} \frac{Z^{3/7}(\phi)}{\phi^2} d\phi$$

**YES, equality holds true when** $\xi_0 = 6.439$ ✓



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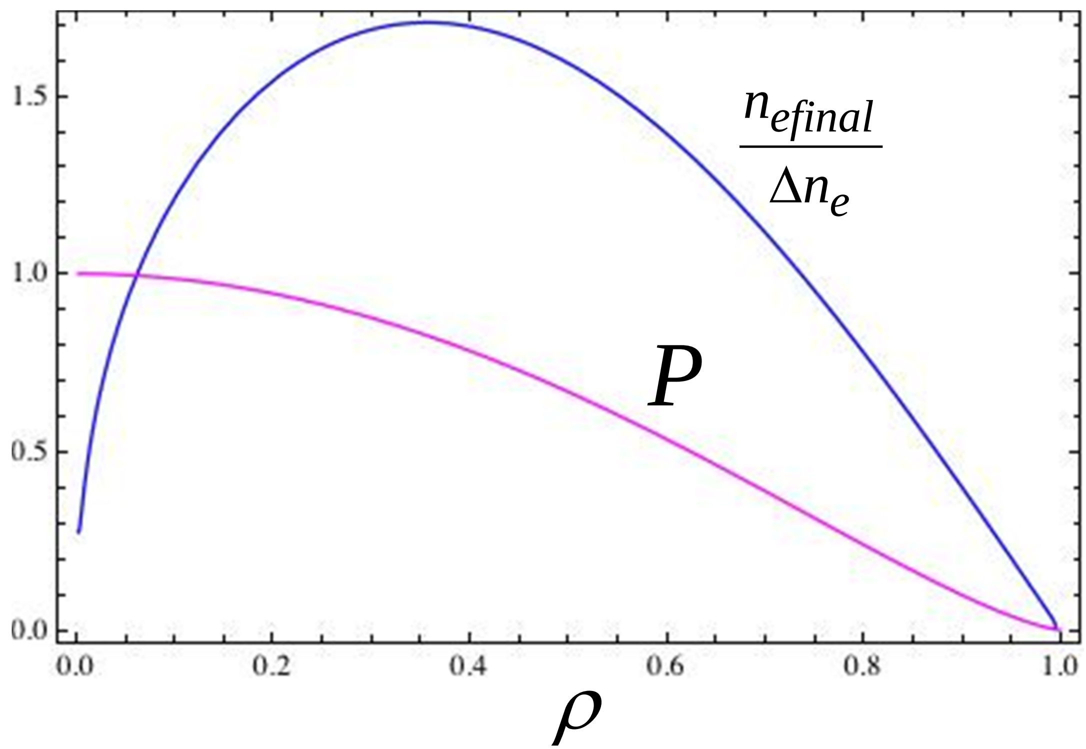
# Plot of Final Density Profile

- **Using a special normalized pressure profile** $P(\rho)=(1-\rho^2)^{7/5}$

$$\longrightarrow \quad F(\rho)/F(0)=1-\left(\frac{25}{14}\rho^{11/7}-\frac{11}{14}\rho^{25/7}\right) \quad \text{with} \quad F(0)=\frac{98}{275}=0.356$$




- **For optimum injection, the added density profile is skewed towards the magnetic axis** $\rho=0$

# Summary and Future Directions

- Developed a 1-D analytic model for the penetration of SPI plume in a plasma which includes plasma cooling by the ablated gas trail
  - two cooling models: kinetic based for neon-D2 and dilution cooling for mainly D2.
  - will compared results with NIMROD that assume dilution cooling with ion Te = Ti
- Publish Z > 1 pellet ablation models and SPI theory
- Improve hot tail RE burst physics model for realistic SPI and pellet injection situations
- Extend SPI model to 2-D geometry important for optically thick gas
- Explore alternative particle injection approaches such as Be shell pellets